Robust and scalable simulation-based inference

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Greek Stochastics - Folegandros 2025

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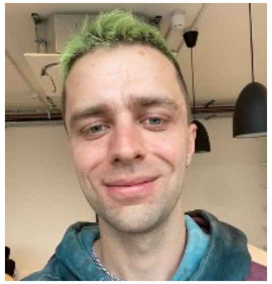

A (slightly biased) introduction to simulation-based inference

Intractable likelihoods

Our data: $y_1, \ldots, y_n \sim \mathbb{Q}$

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Unknown data-generating process defined on the data-space $\mathscr{X}$.

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Our job is to recover θ^*

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Maximum likelihood:

$$\hat{\theta}_n := \arg\max_{\theta \in \Theta} \prod_{i=1}^{n} p(y_i \,|\, \theta)$$

Bayesian inference:

$$p(\theta \,|\, y_1, \ldots, y_n) \propto \prod_{i=1}^{n} p(y_i \,|\, \theta) p(\theta)$$

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Simulation-based inference (SBI)

- A simulator $(\mathbb{U}, G_\theta)$ such that a draw from $\mathbb{P}_\theta$ can be obtained as:

$$x_i = G_\theta(u_i)$$

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(generator)

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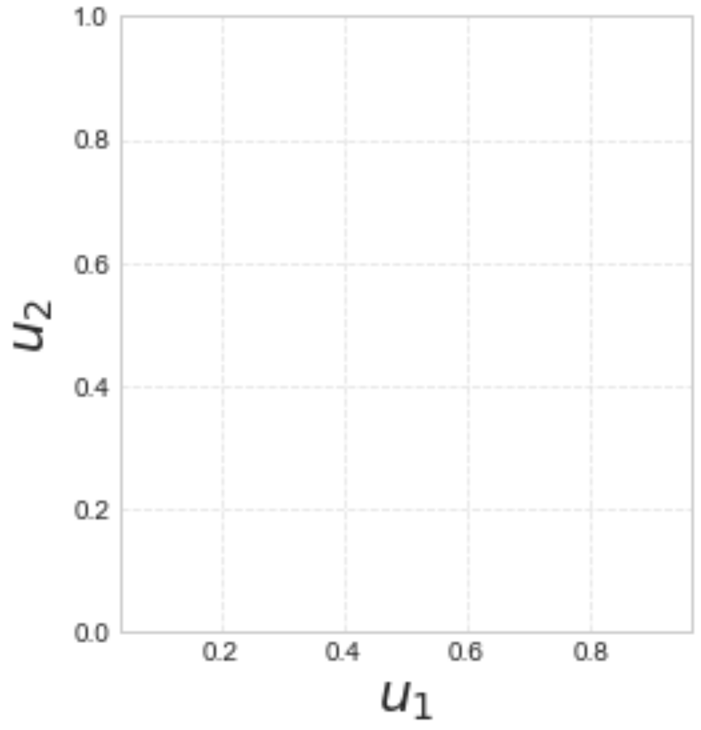
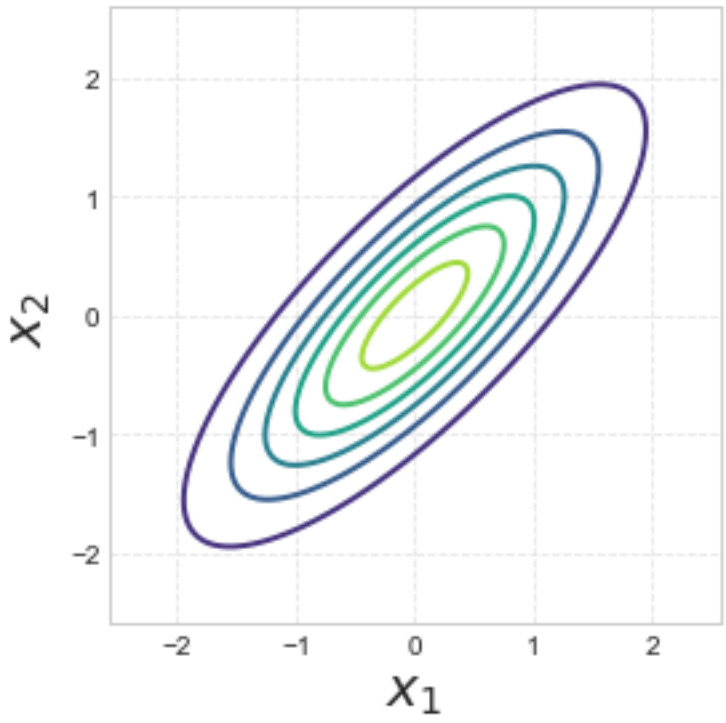
Simulation-based inference: Inference using simulated data to replace evaluations of the likelihood!

A trivial simulator for Gaussians

- $\mathbb{P}_\theta := \mathcal{N}(\mu, \Sigma)$, $\quad u_i = (u_{i1}, u_{i2})^\top, \quad u_{i1}, u_{i2} \sim \text{Unif}(0,1) \quad G_\theta(u) = \mu + L \begin{pmatrix} \Phi^{-1}(u_{i,1}) \\ \Phi^{-1}(u_{i2}) \end{pmatrix}$





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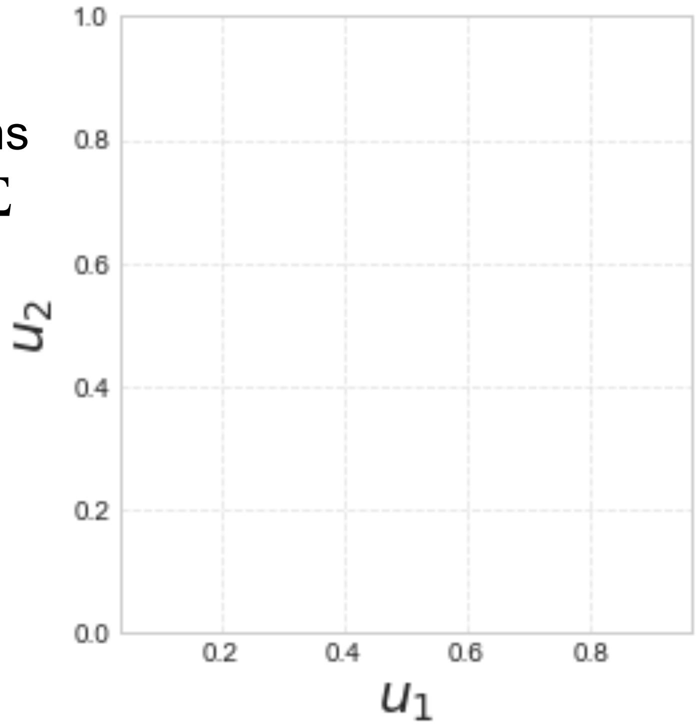
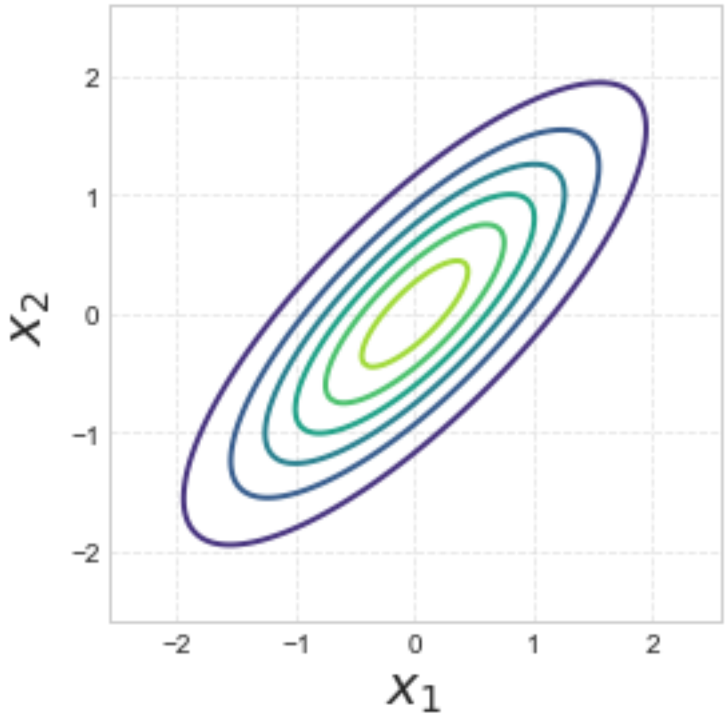
i.e. θ contains both μ and Σ





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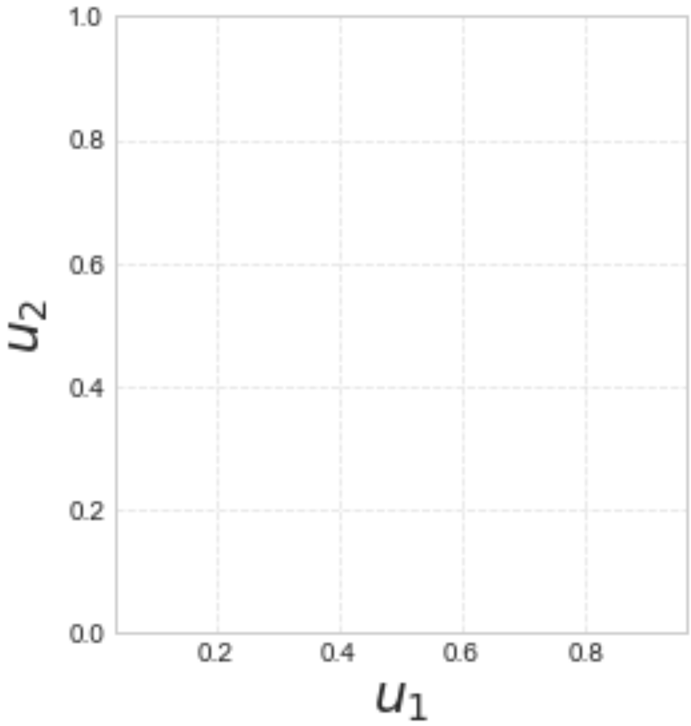

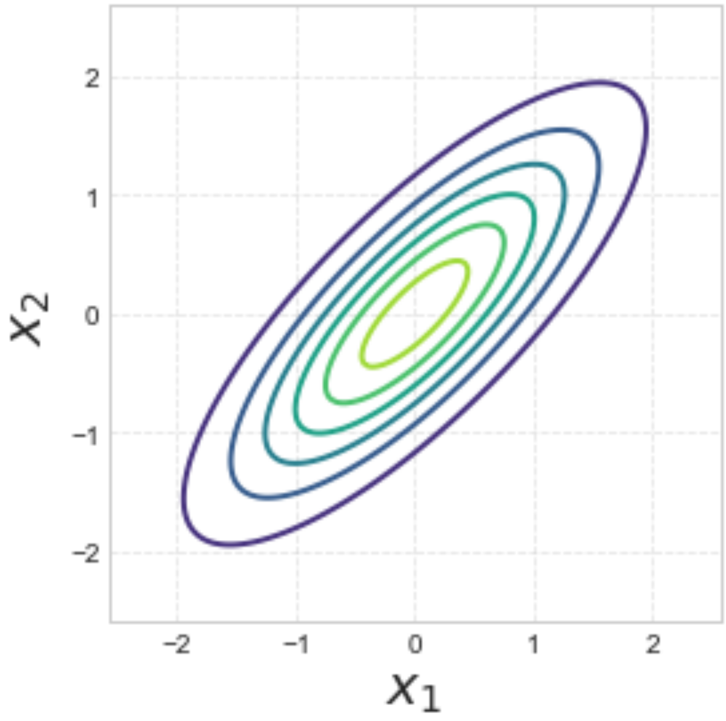

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Inverse CDF of standard Gaussian!

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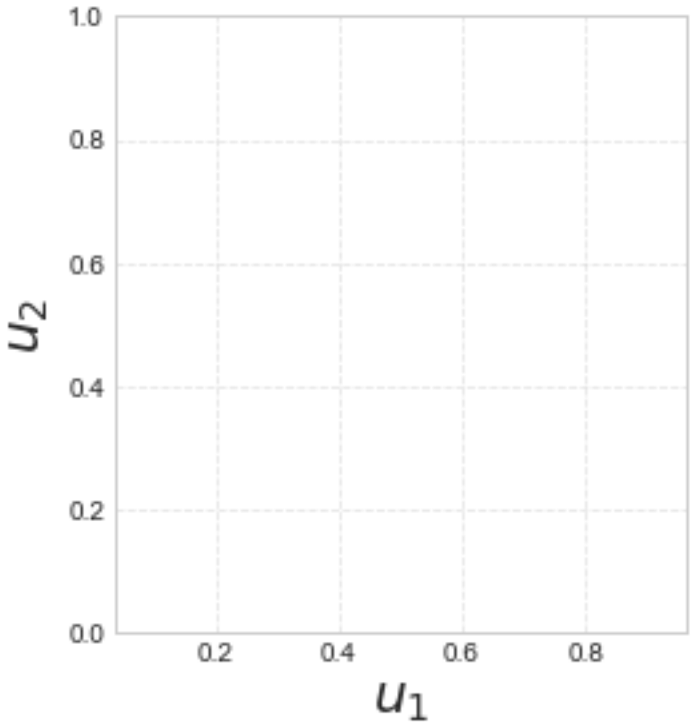

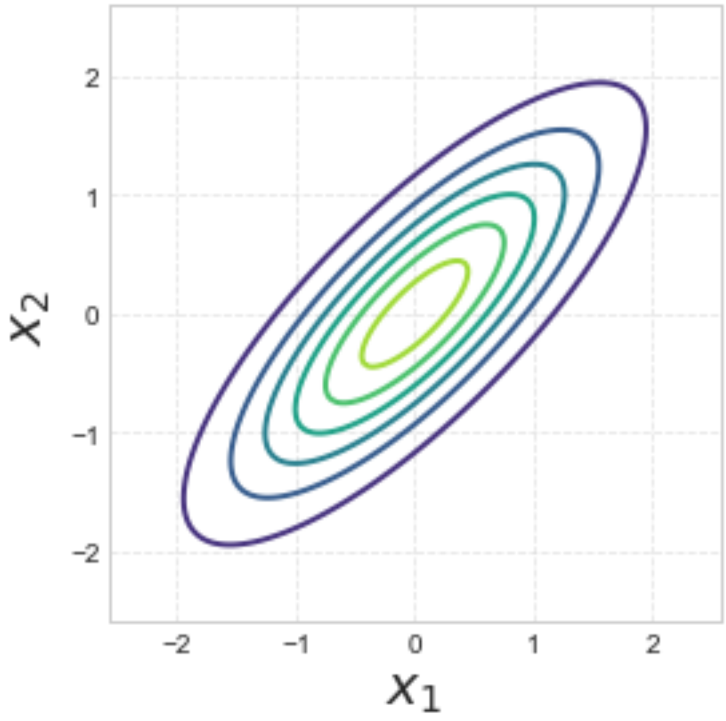

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Cholesky!

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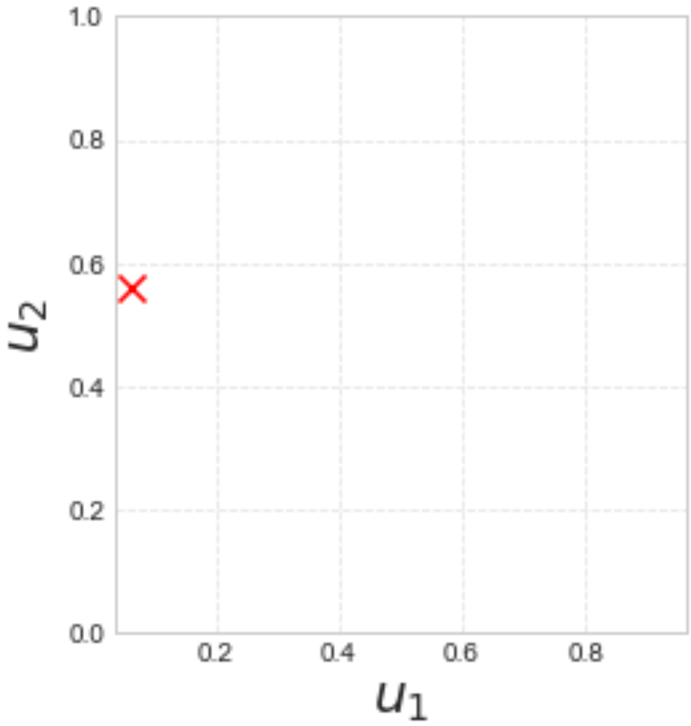

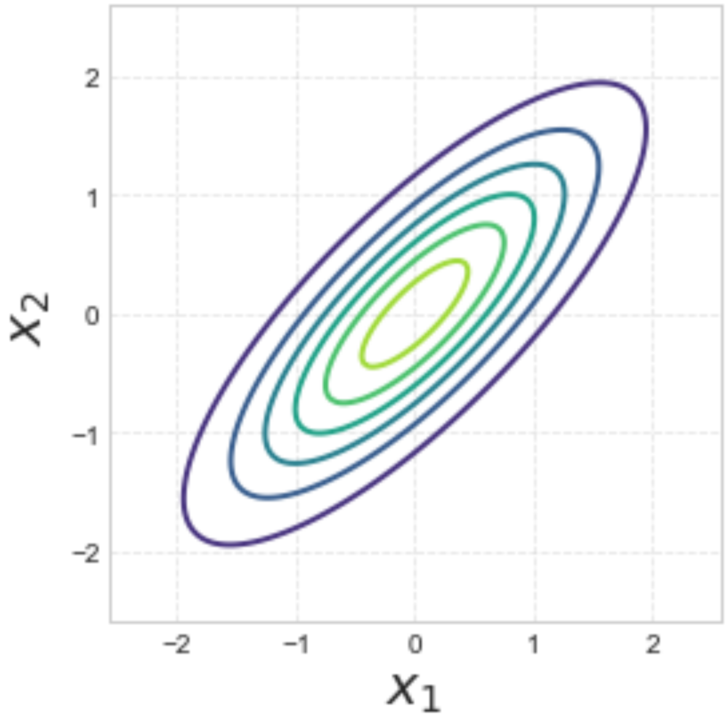

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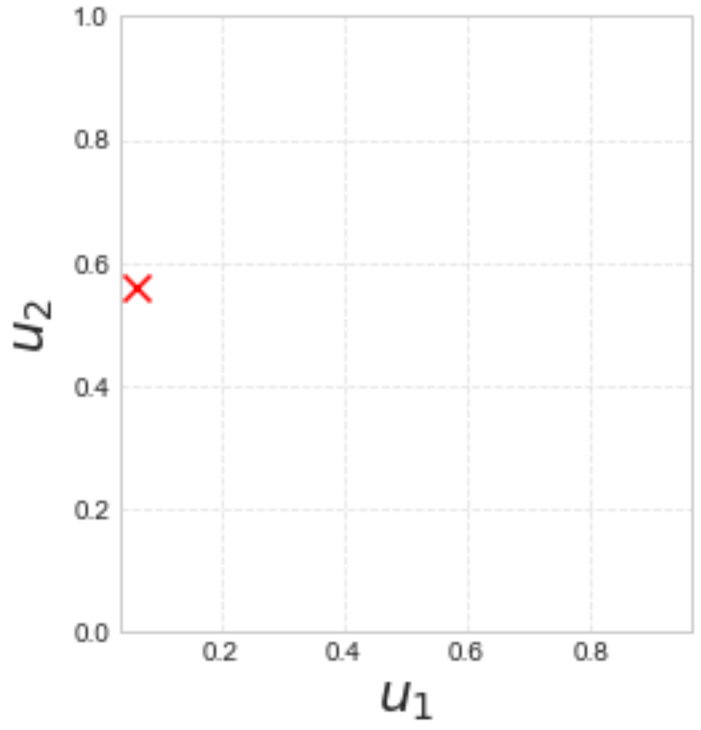

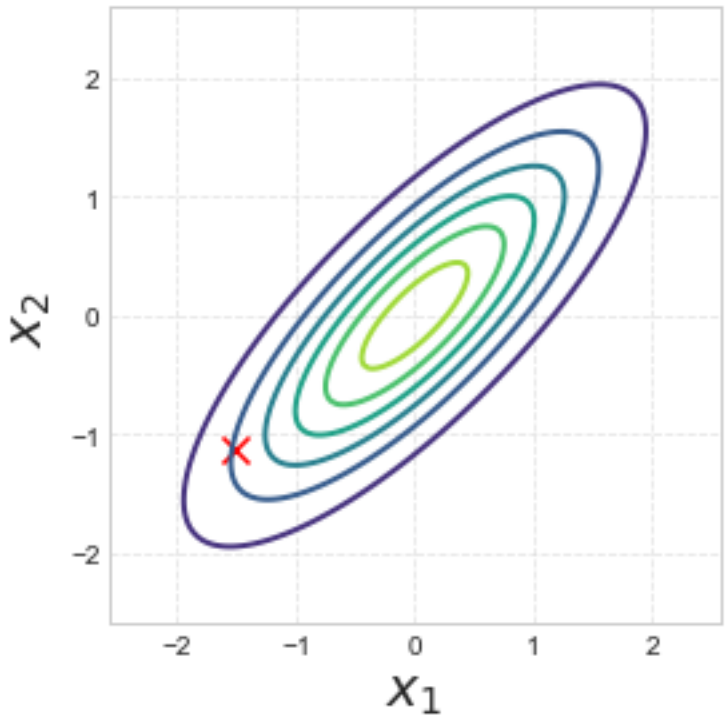

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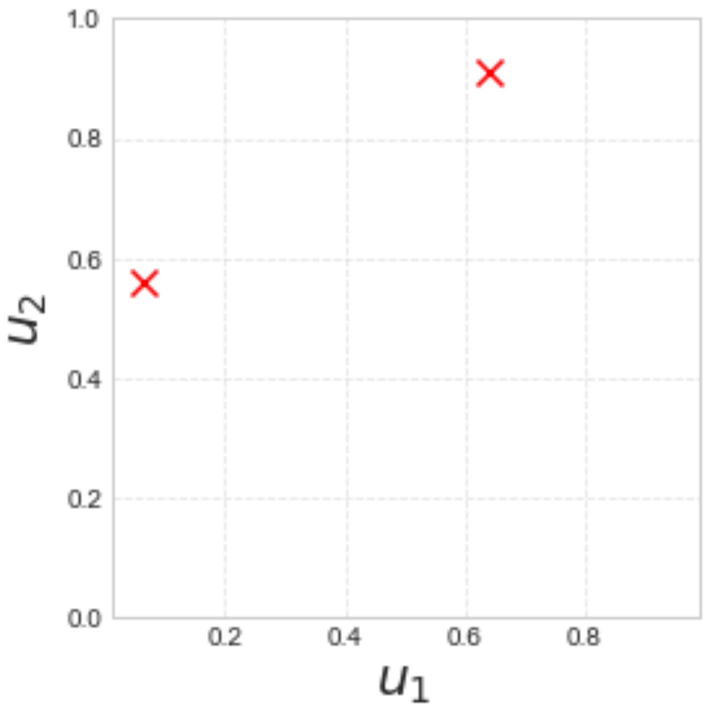

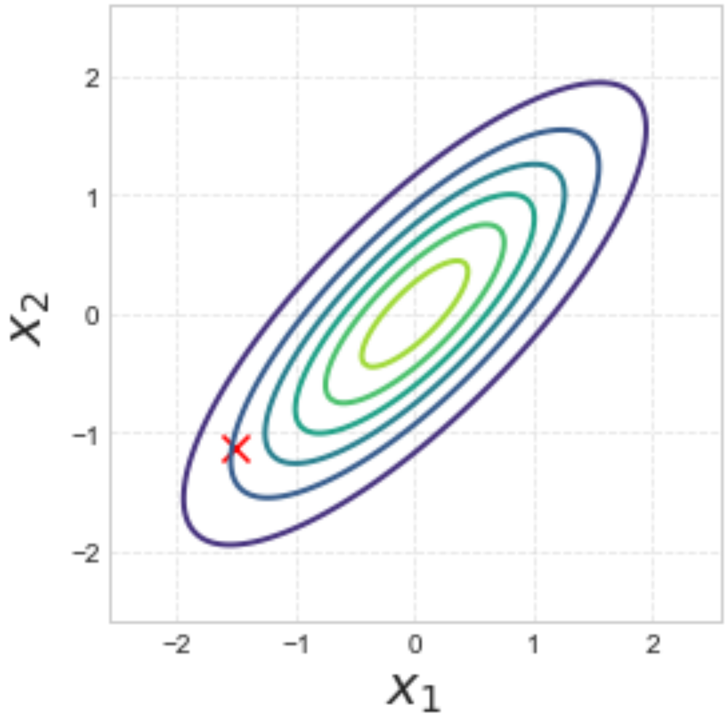

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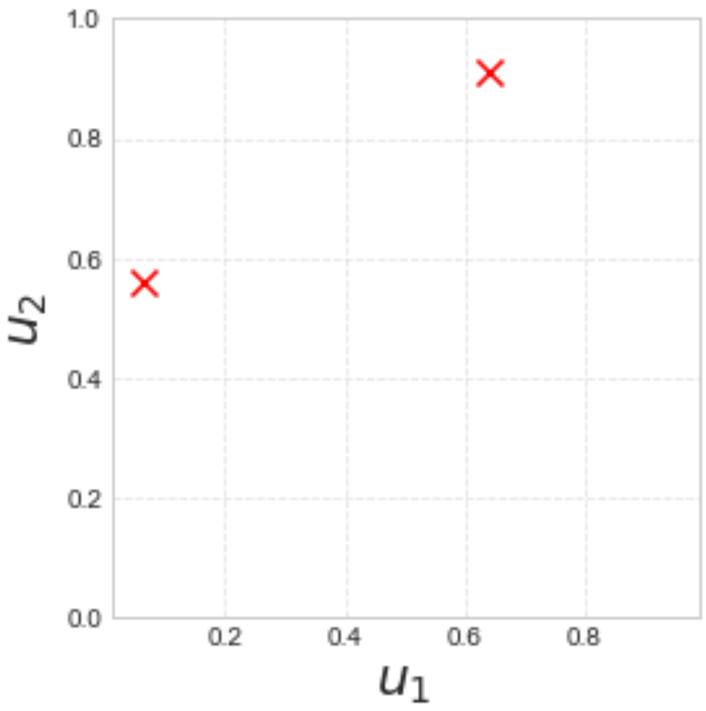

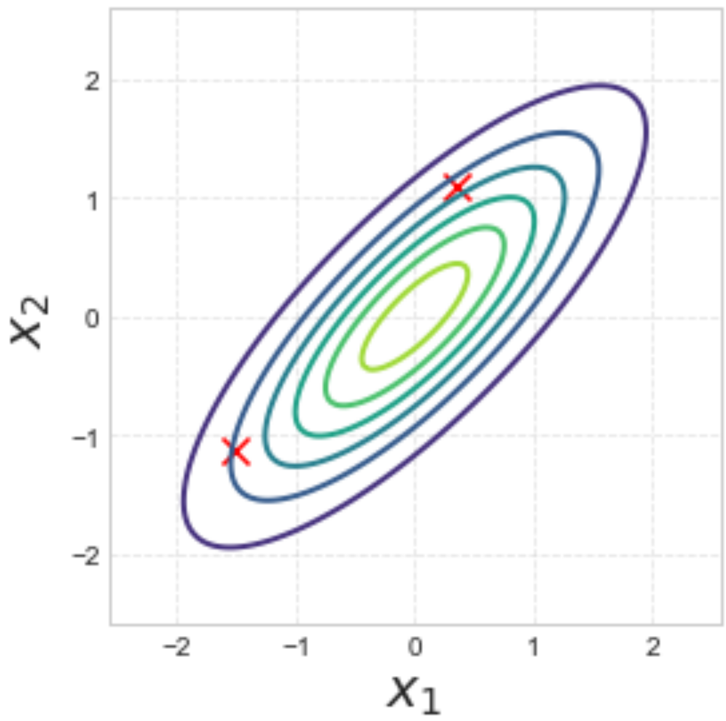

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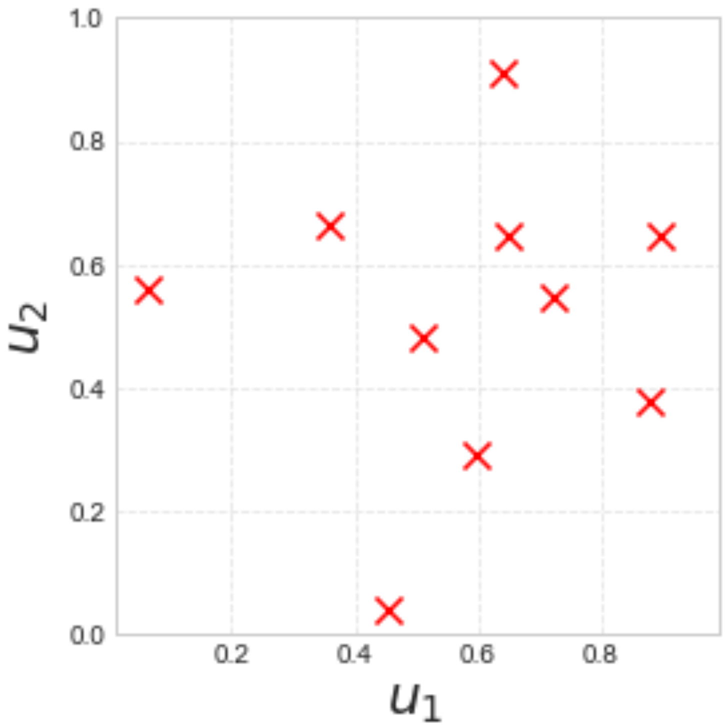

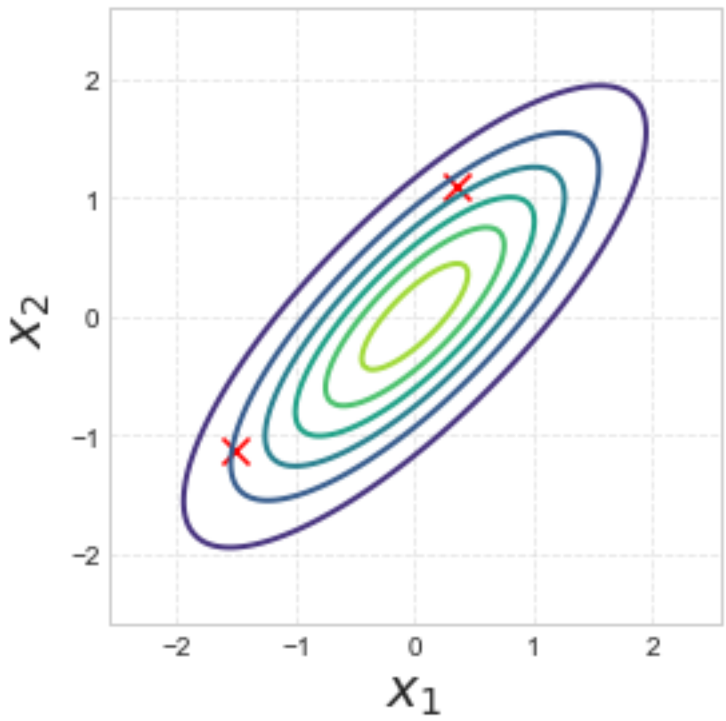

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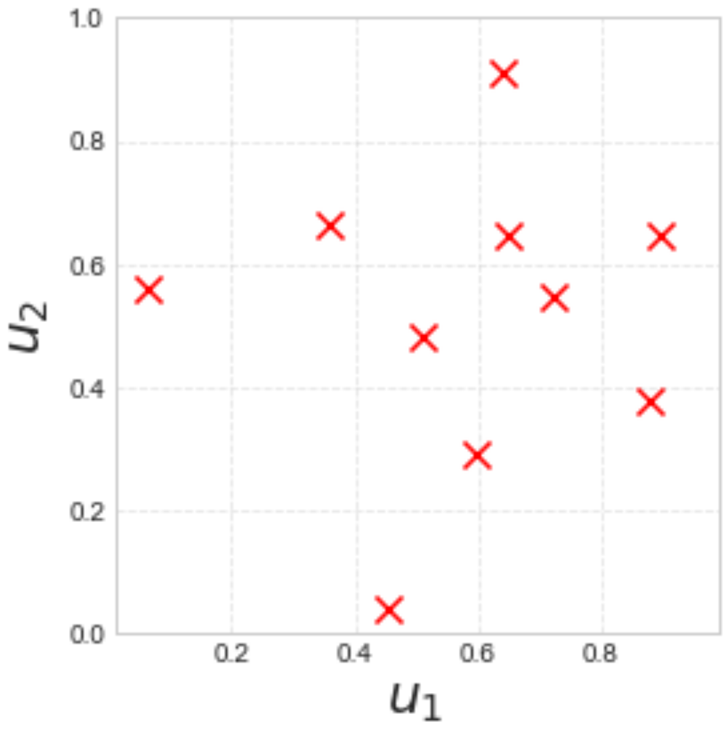

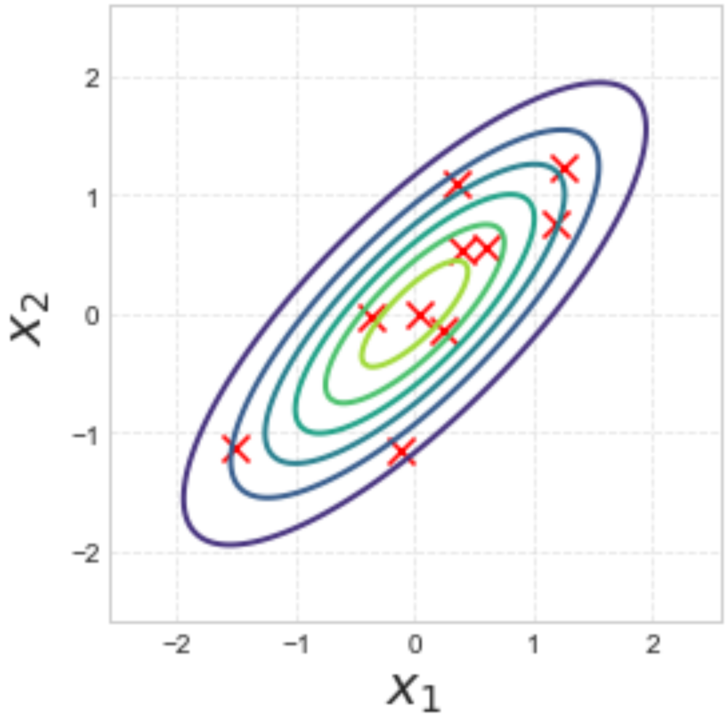

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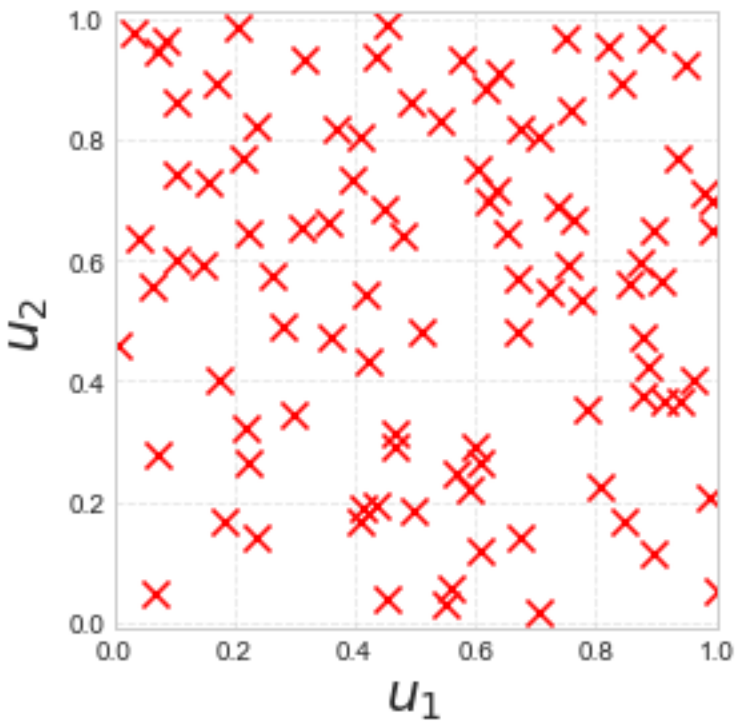

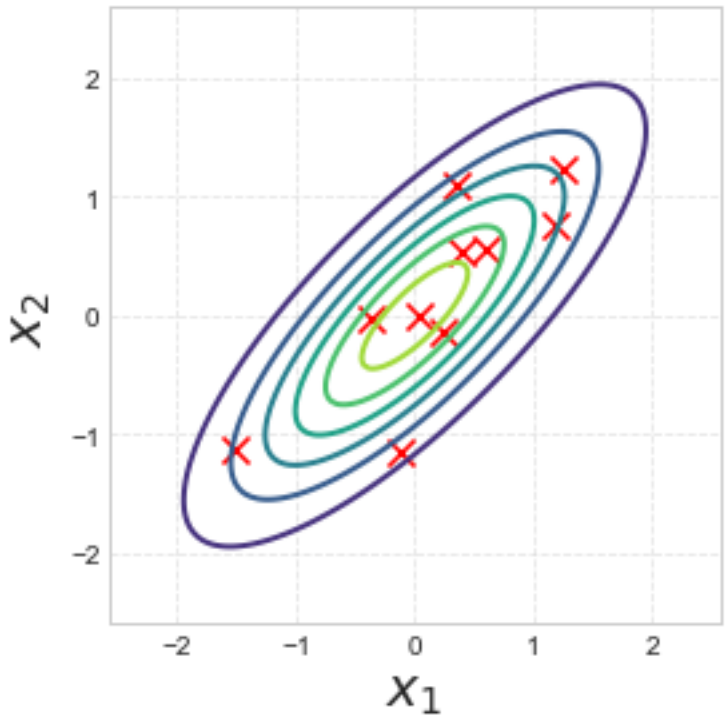

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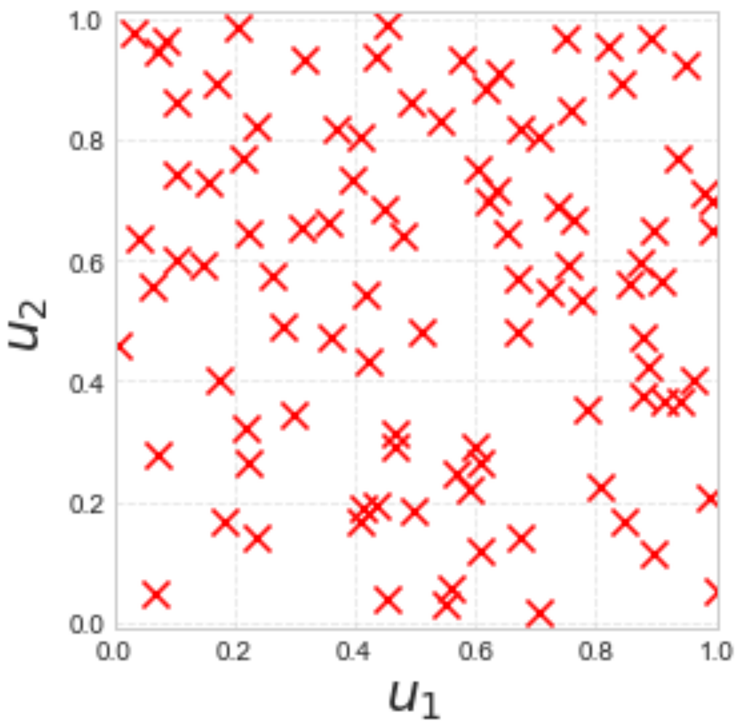

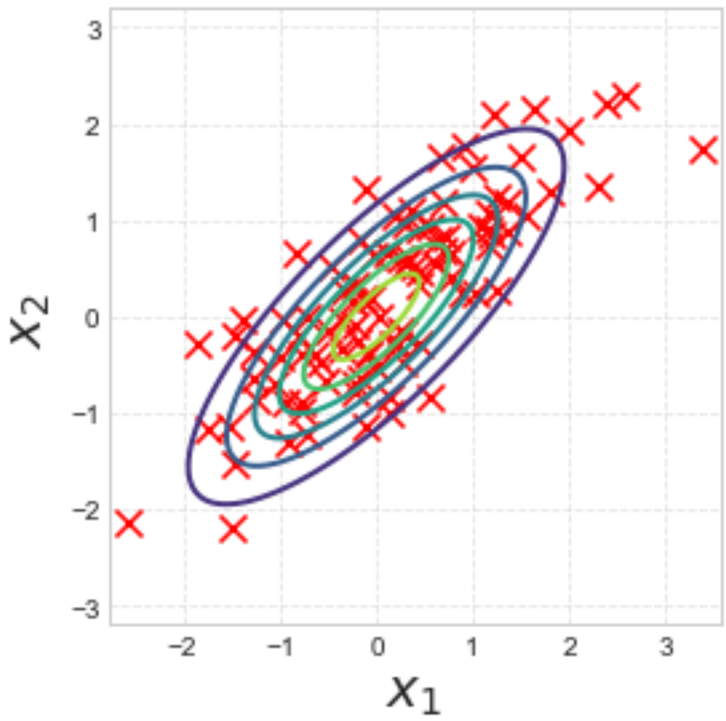

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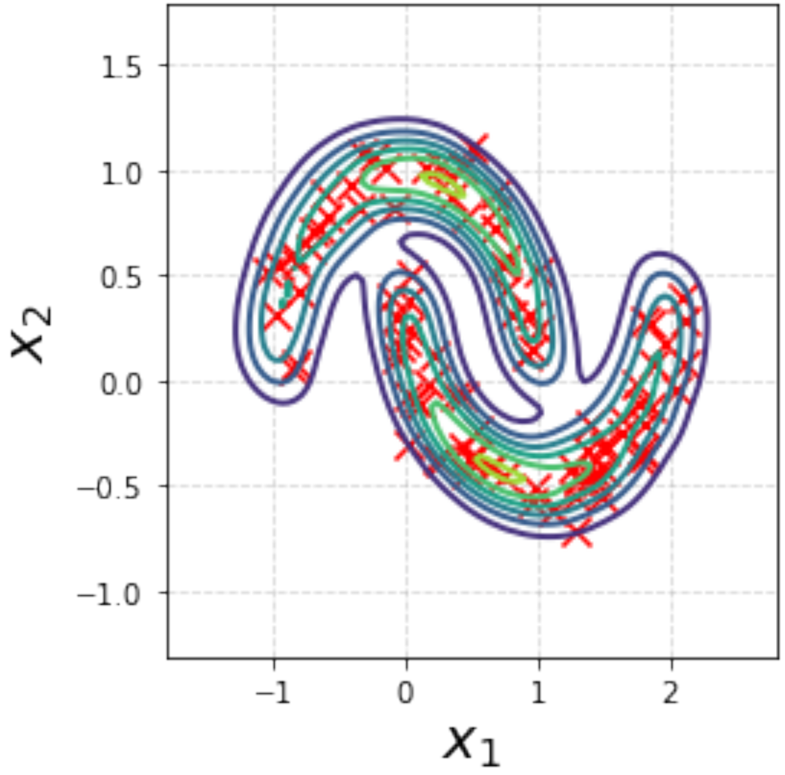
Some slightly less trivial simulators….

- We can create all sorts of more complex simulators by increasing the complexity of the G_θ map.



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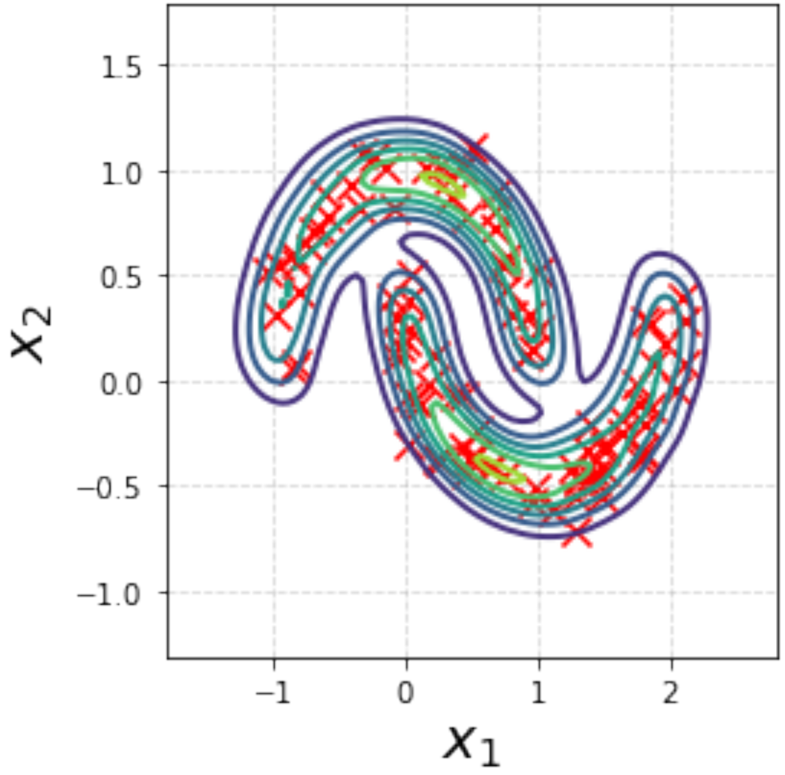

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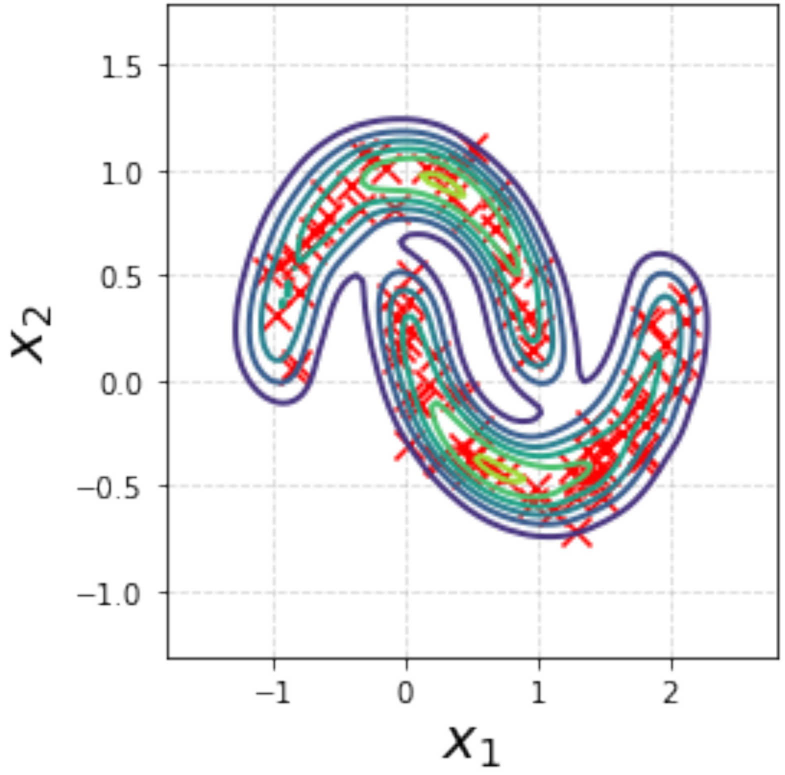
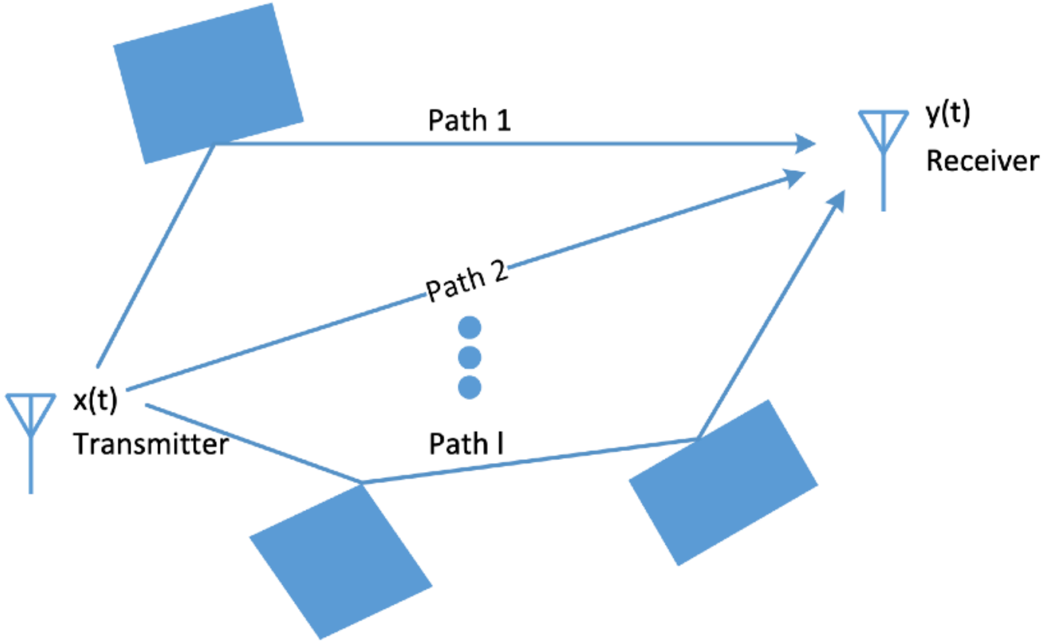
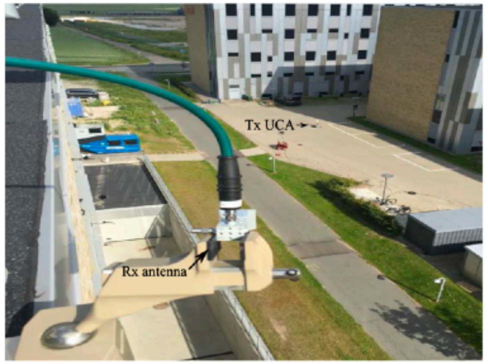
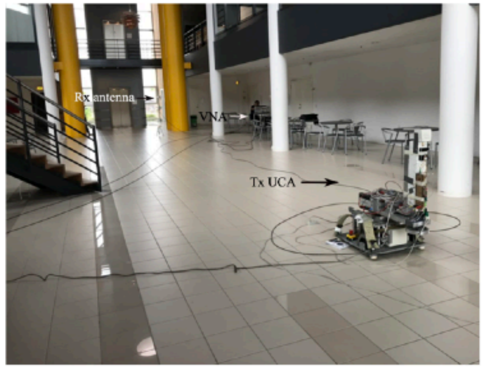
- SBI often works with simulators **carefully crafted by scientists and engineers**. These simulators are hence implementations of complex mathematical models of the phenomena being studied.



Simulators in telecommunications engineering







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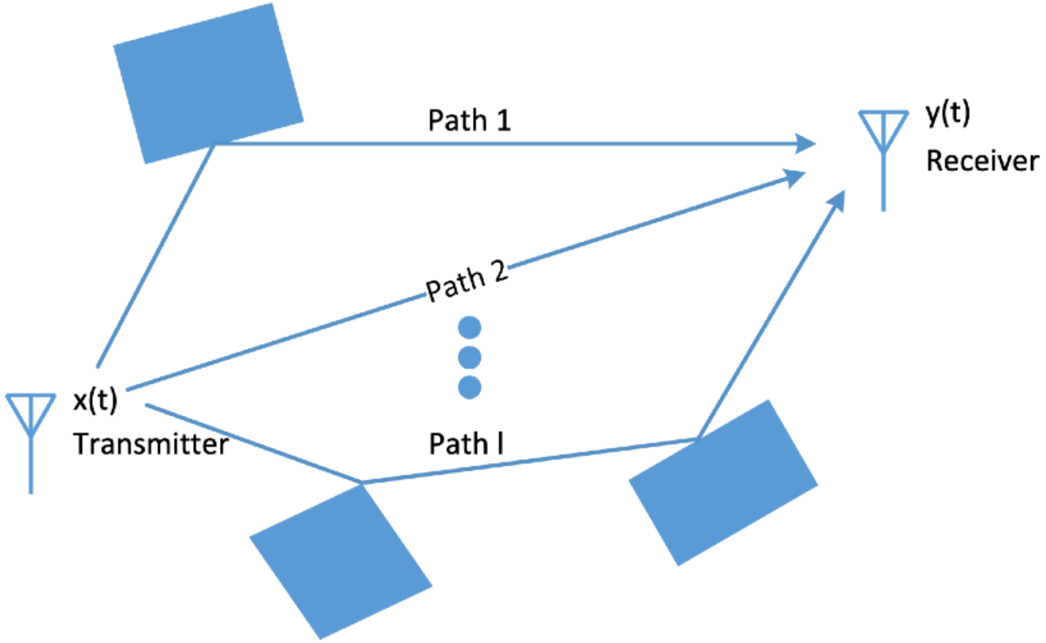

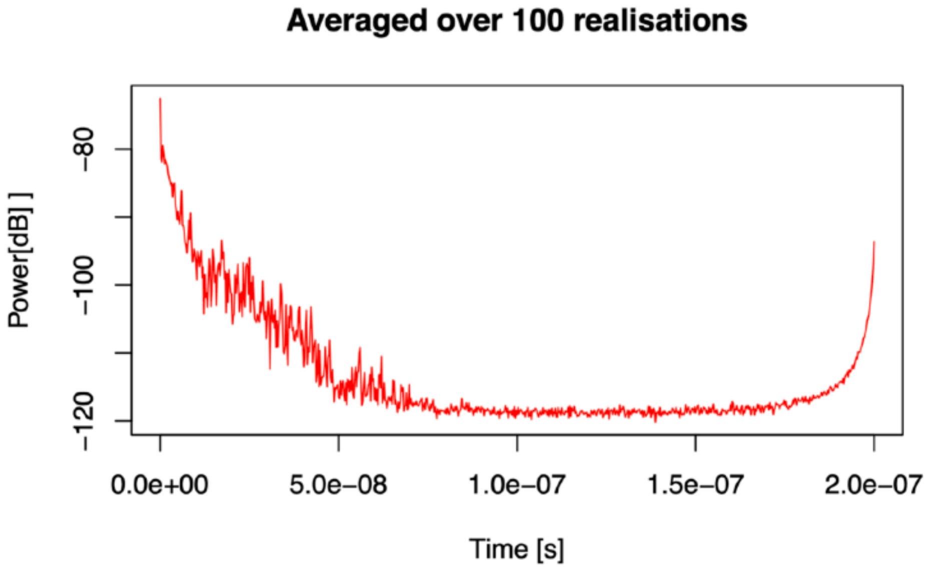

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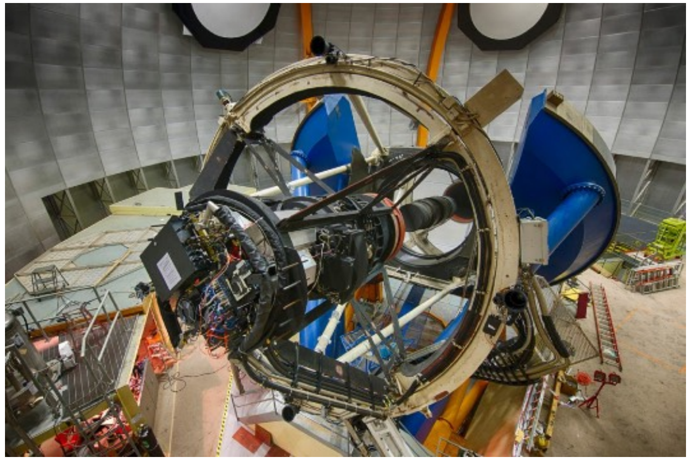
Simulators in cosmology

(+ ≈ 400 scientists from 25 institutions in 7 countries)



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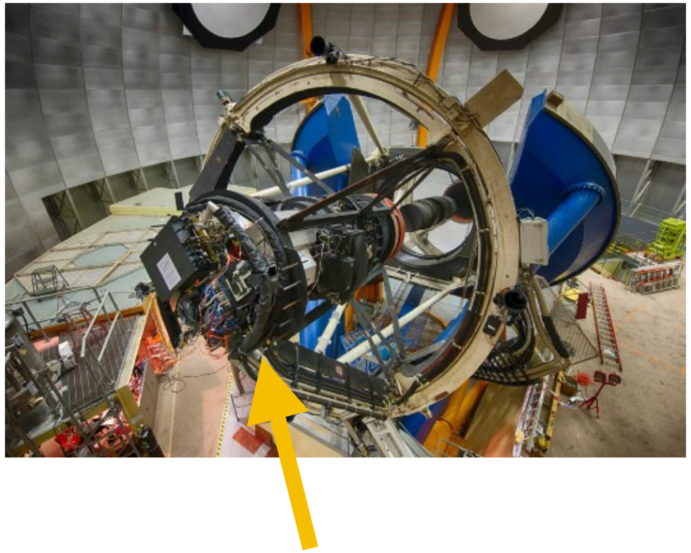

The Dark energy survey camera!

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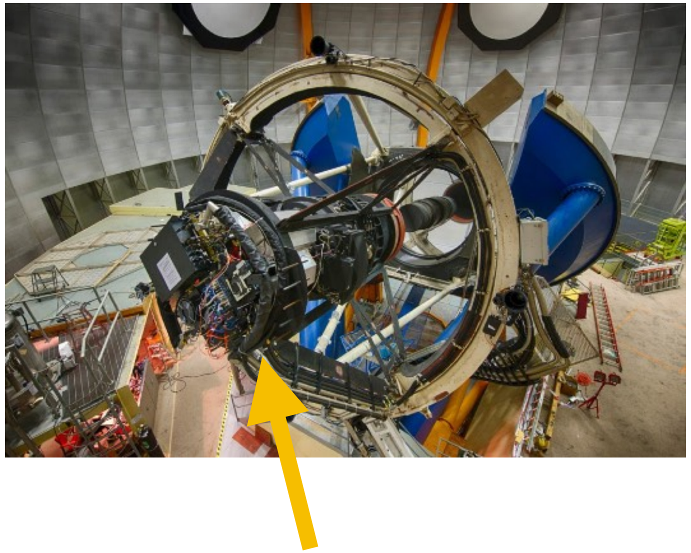

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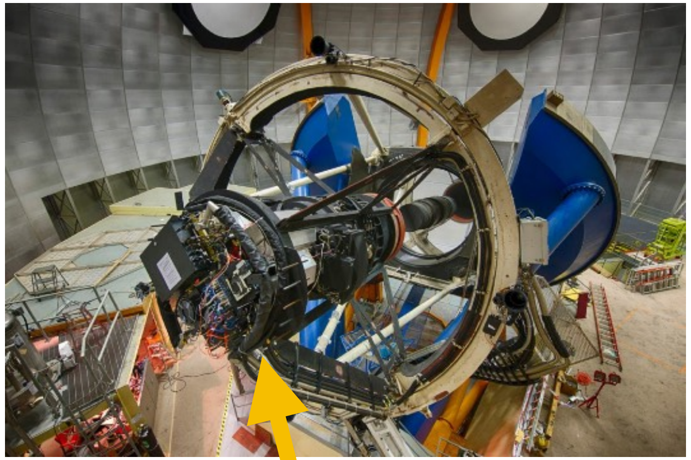

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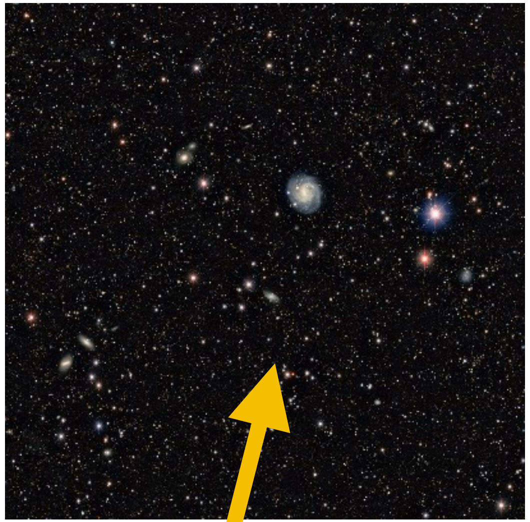

Data collected through the Dark energy survey camera

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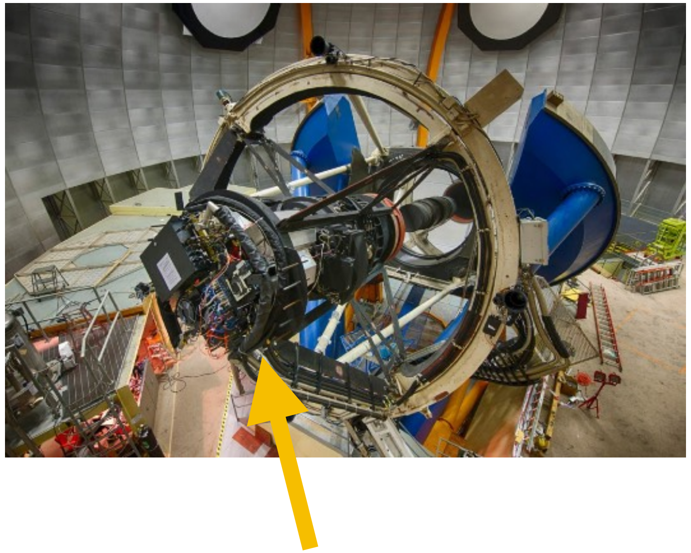

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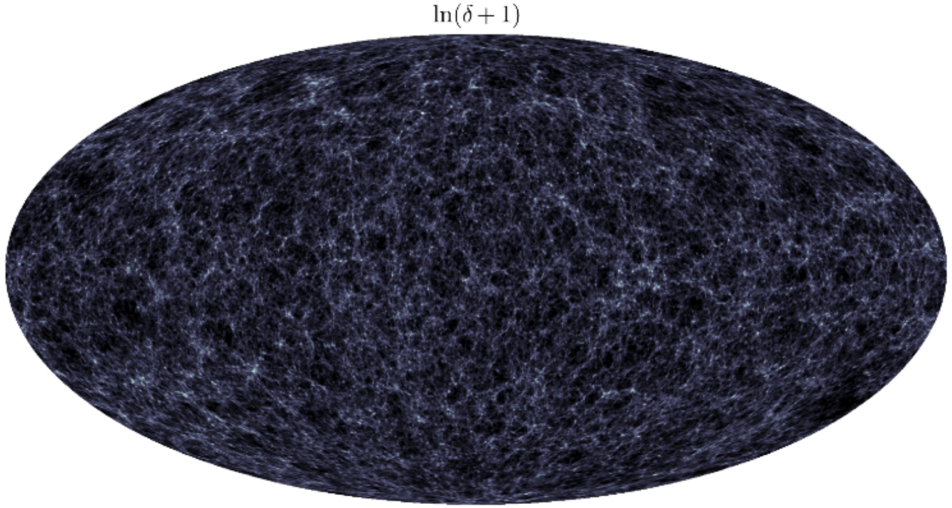

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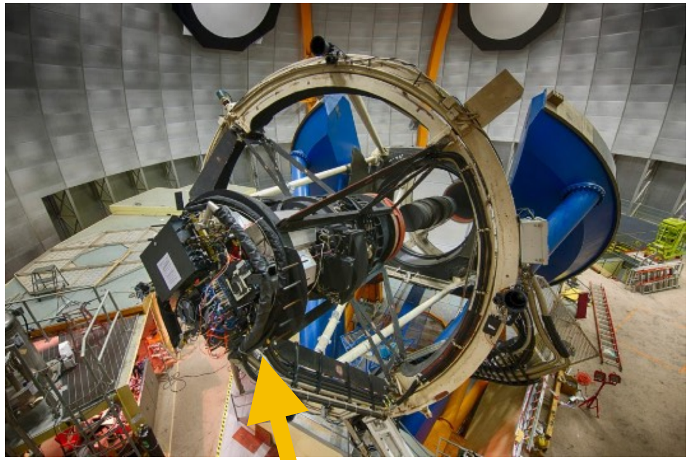

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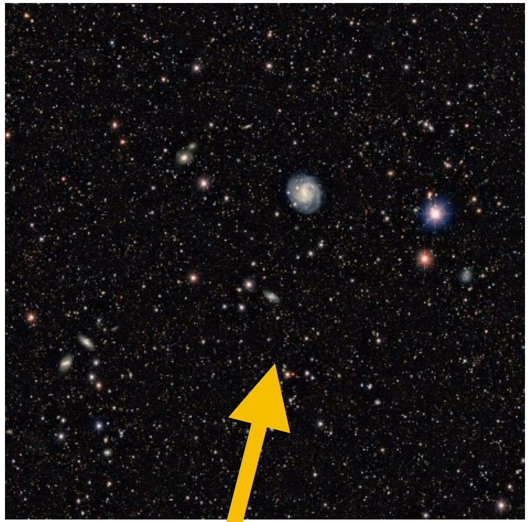

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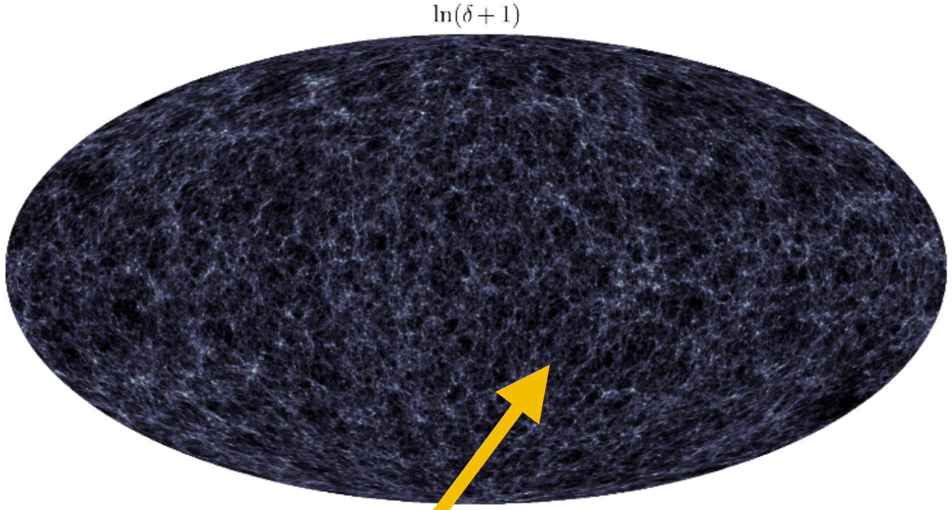

'Gower Street simulation' run by Niall and colleagues at UCL Physics

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Simulators in the sciences and beyond

Particle Physics (CERN)

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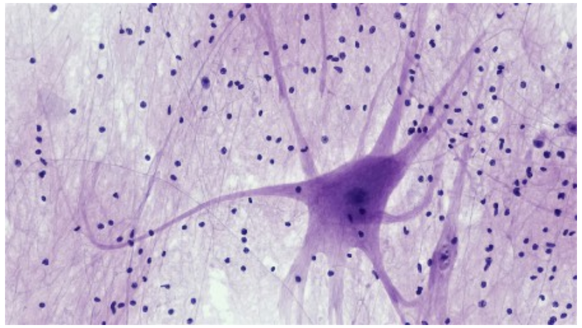

Neuroscience

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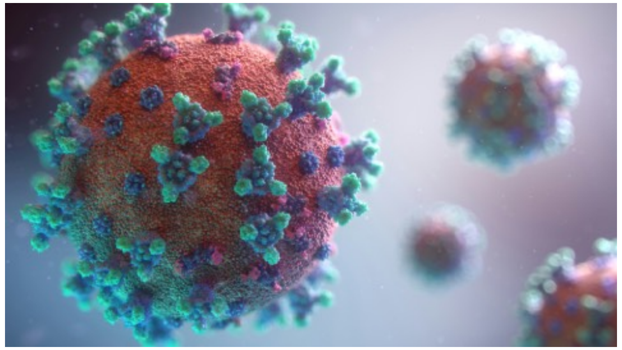
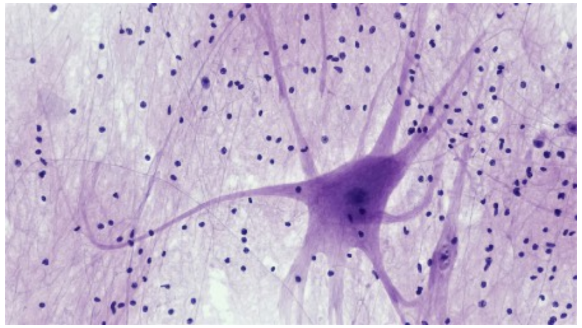


Epidemiology



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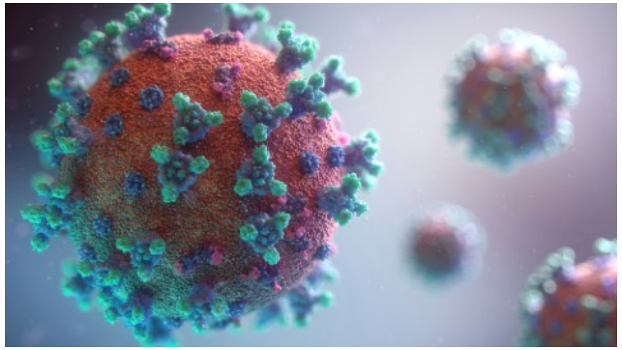

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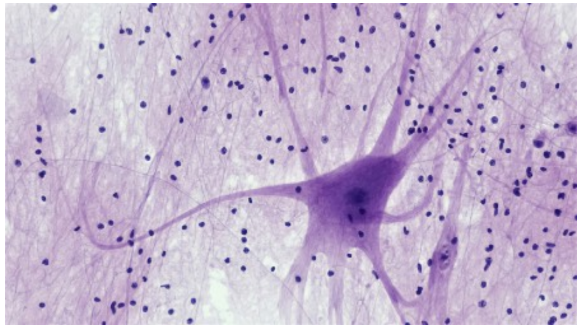

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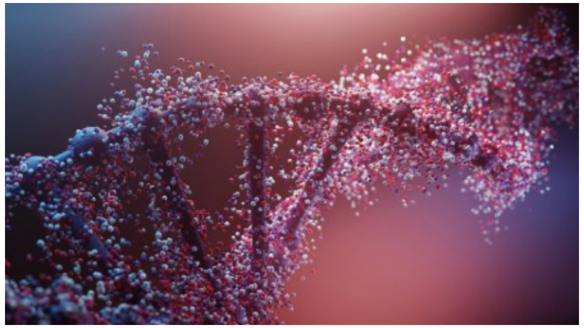

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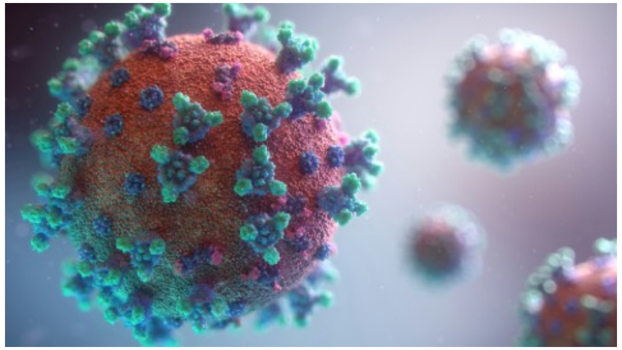

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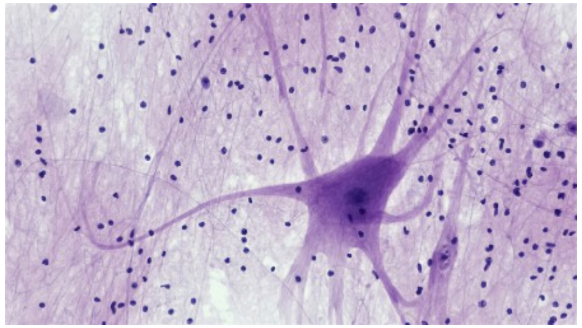

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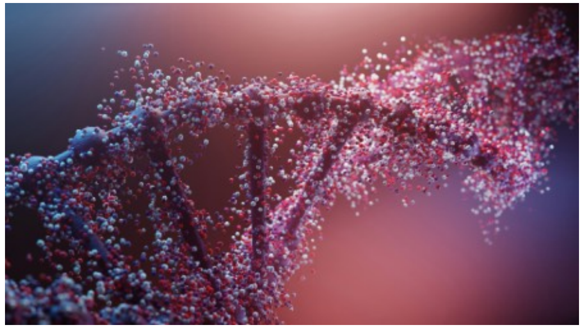

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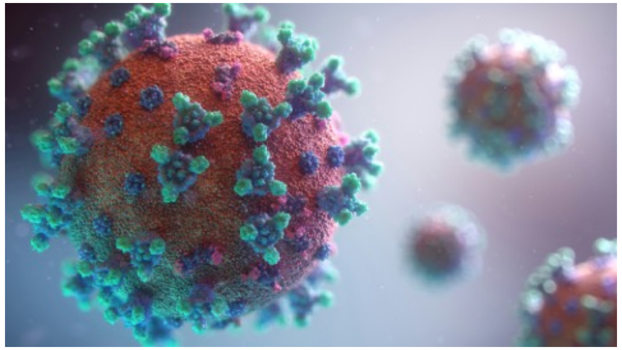

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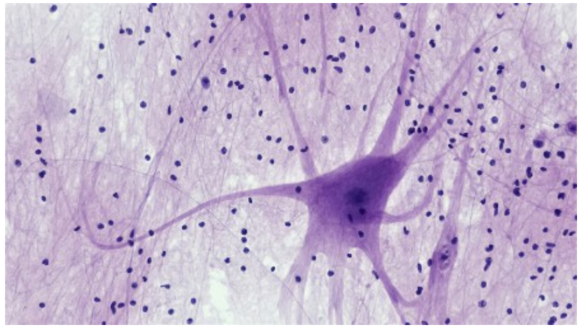

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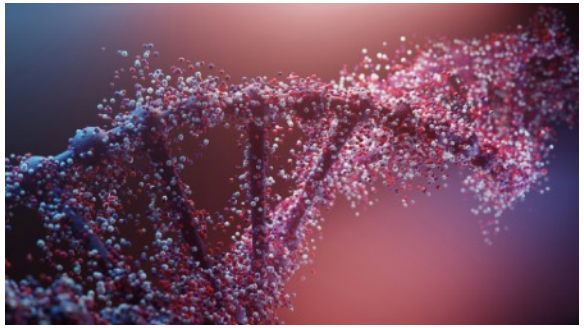
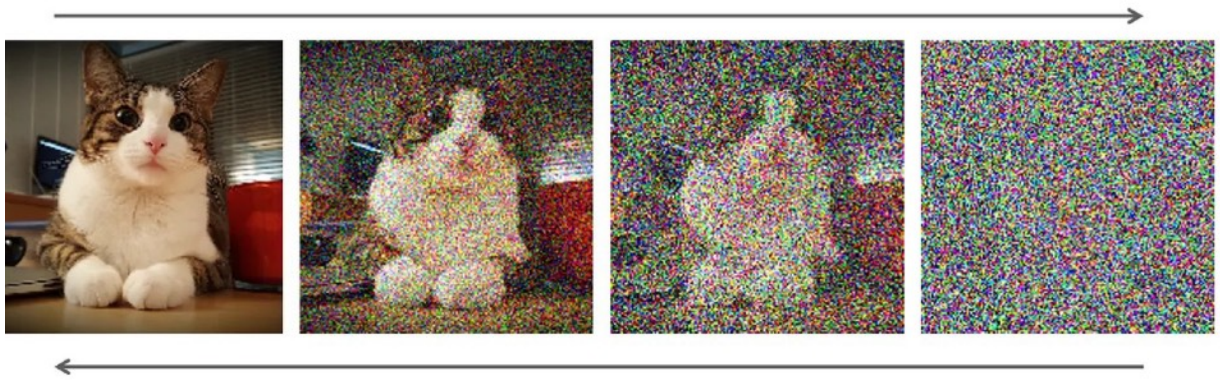


Genomics

https://simulation-based-inference.org/

Clarifying terminology

- **Are diffusion models simulators?**



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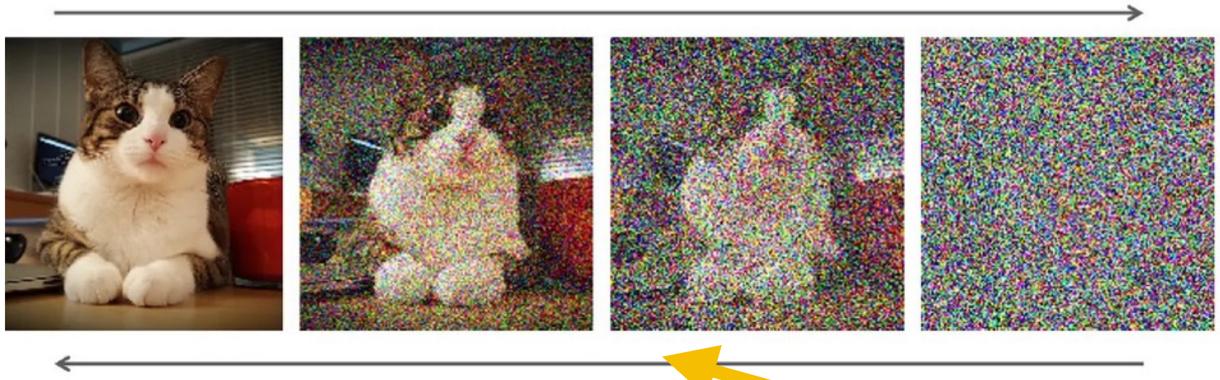

Yes, you can think of this process as defining a simulator!

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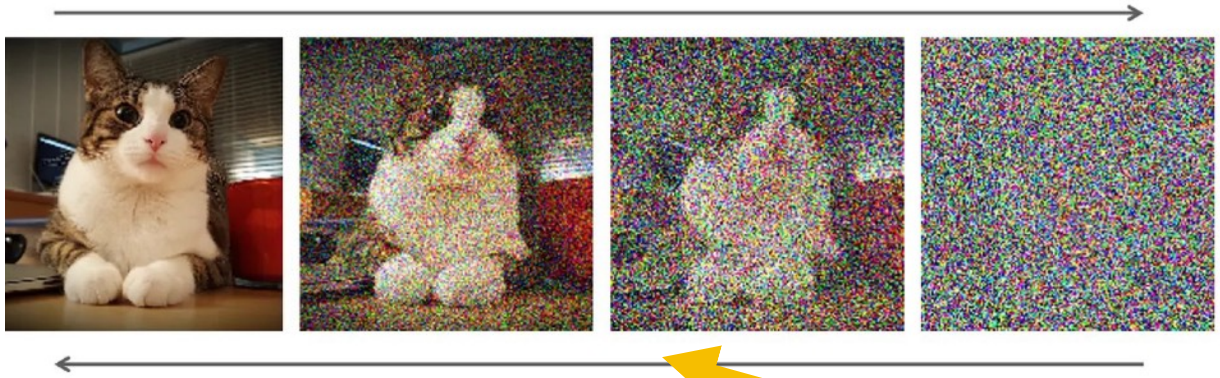

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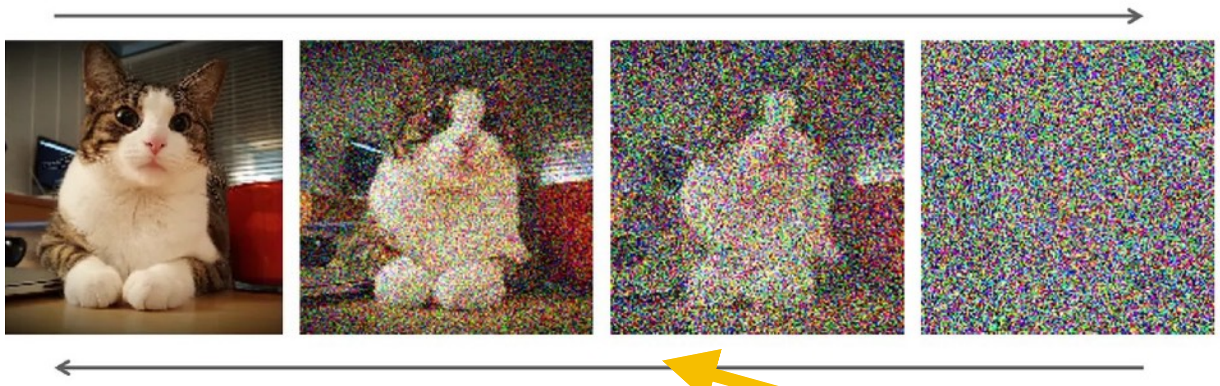

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No! In SBI, we typically have **scientifically meaningful simulators** where t**he parameter** θ **can be interpreted.** We therefore really care about estimating it and providing **uncertainty estimates**!

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Any Questions?

What is coming up

- Basic methods:

Minimum distance estimation

Approximate Bayesian Computation

Neural simulation-based inference

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- Discussion of the main challenges in SBI.

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- Basic methods:

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- Discussion of the main challenges in SBI.
- Some illustrations of recent advances:

Hikida, Y., Bharti, A., Jeffrey, N. & **Briol, F-X** (2025). Multilevel neural simulation-based inference. arXiv:2506.06087 (to appear at NeurIPS?).

Bharti, A., Huang, D., Kaski, S., & **Briol, F.-X.** (2025). Cost-aware simulation-based inference. International Conference on Artificial Intelligence and Statistics, 28–36.

Dellaporta, C., Knoblauch, J., Damoulas, T. & **Briol, F-X** (2022). Robust Bayesian inference for simulator-based models via the MMD posterior bootstrap. AISTATS, 943-970. Best paper award.

Minimum Distance Estimation

Minimum Distance Estimation

(i.e. how to be a frequentist in SBI…)

The method of moments

- Model: $\{\mathbb{P}_\theta\}_{\theta \in \Theta}$, Data-generating process: $\mathbb{Q}$, Data: $y_1, \ldots, y_n$

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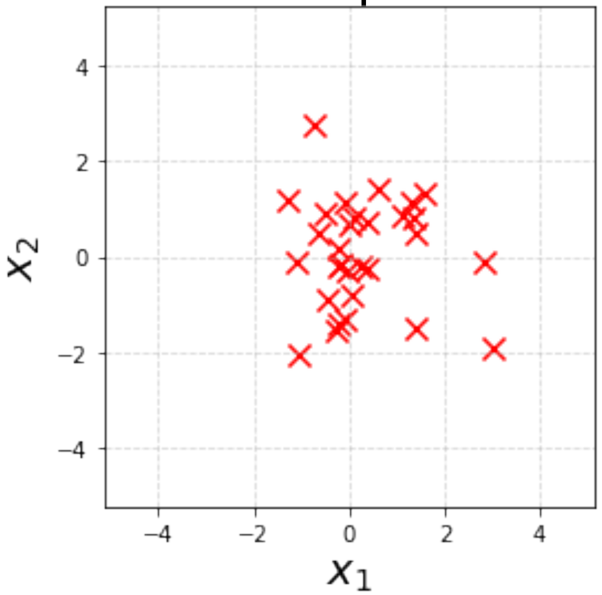

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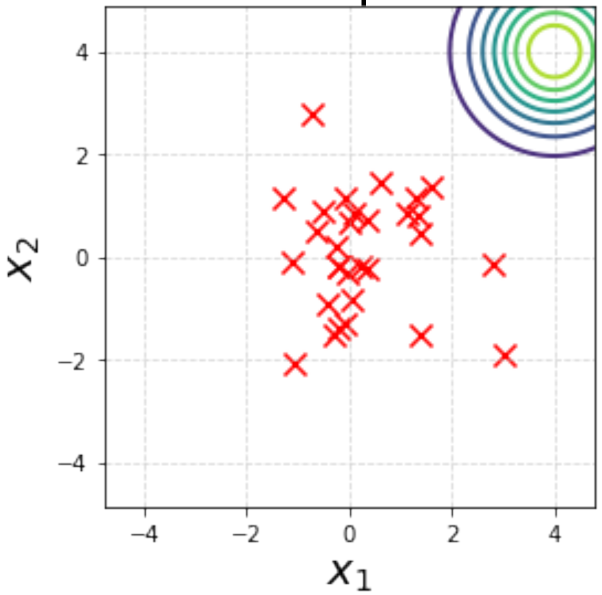

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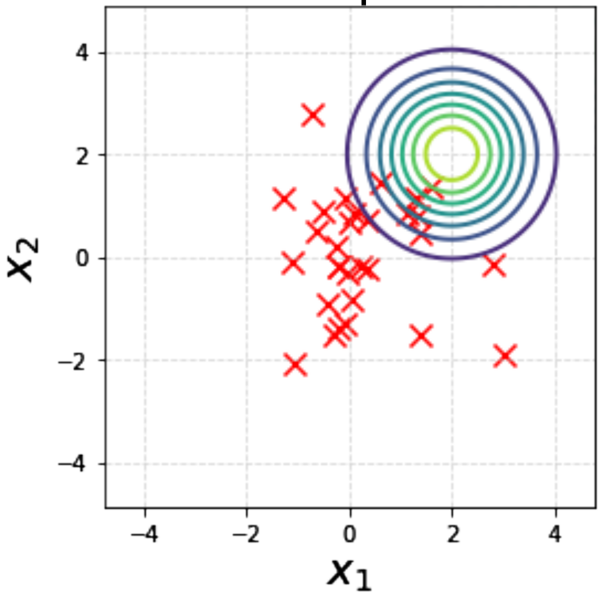

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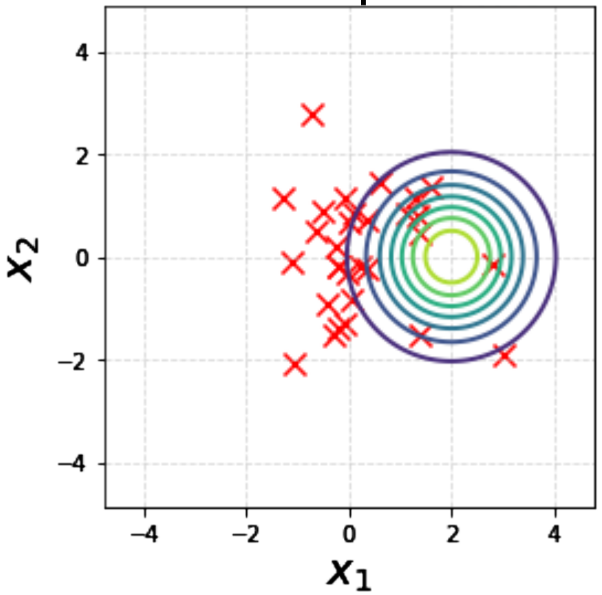

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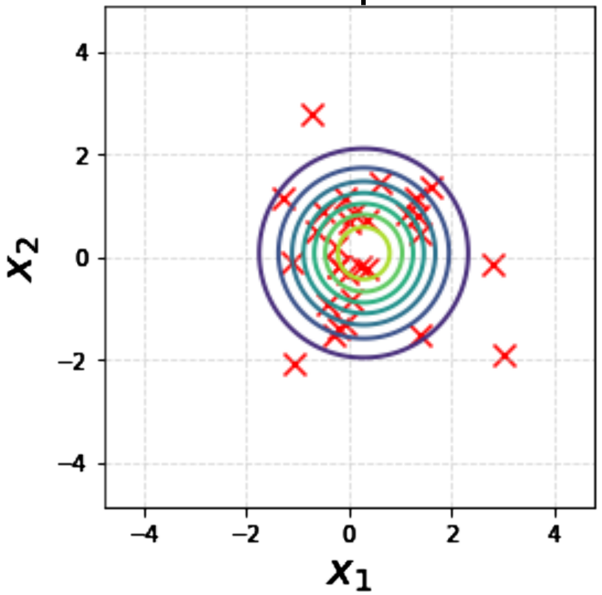

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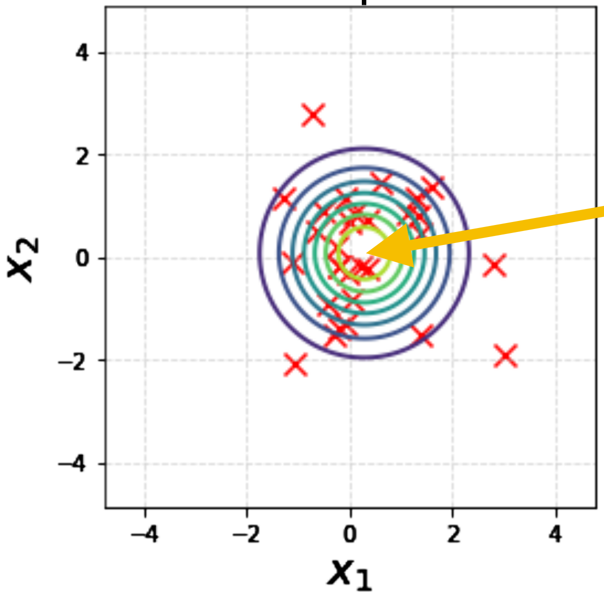

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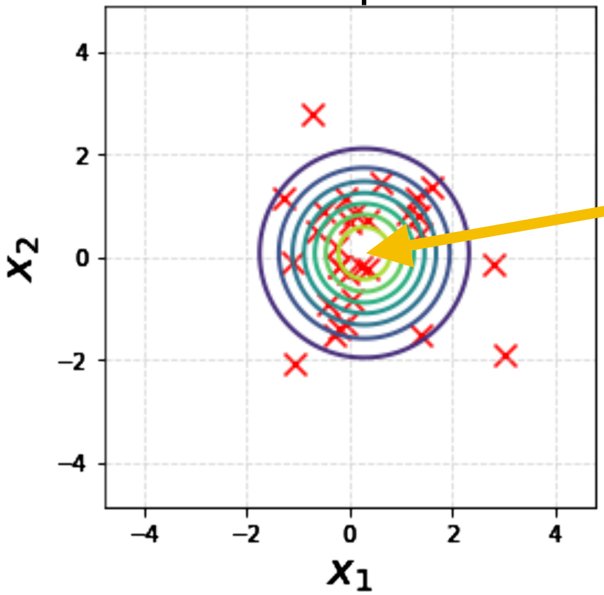

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Note: For more complex models, we may also want to compare higher moments…

The method of <u>simulated</u> moments

- **Problem:** We work with simulators, and so we can't necessarily compute the mean!

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???

- **Method of simulated moments:** We repeat the method of moments, but we simulate at each iteration!

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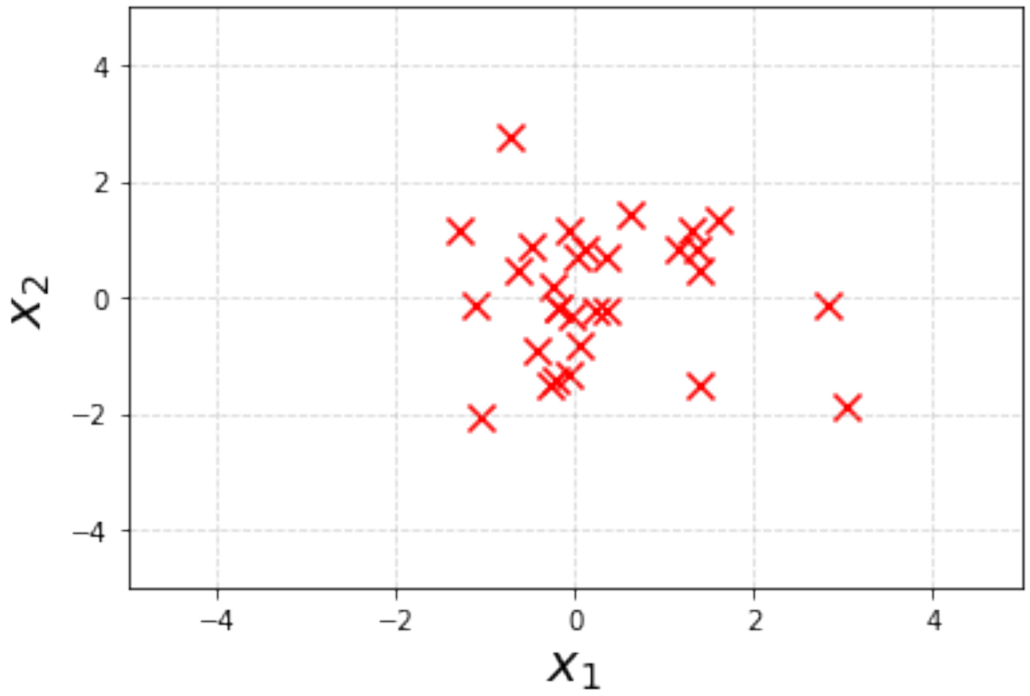

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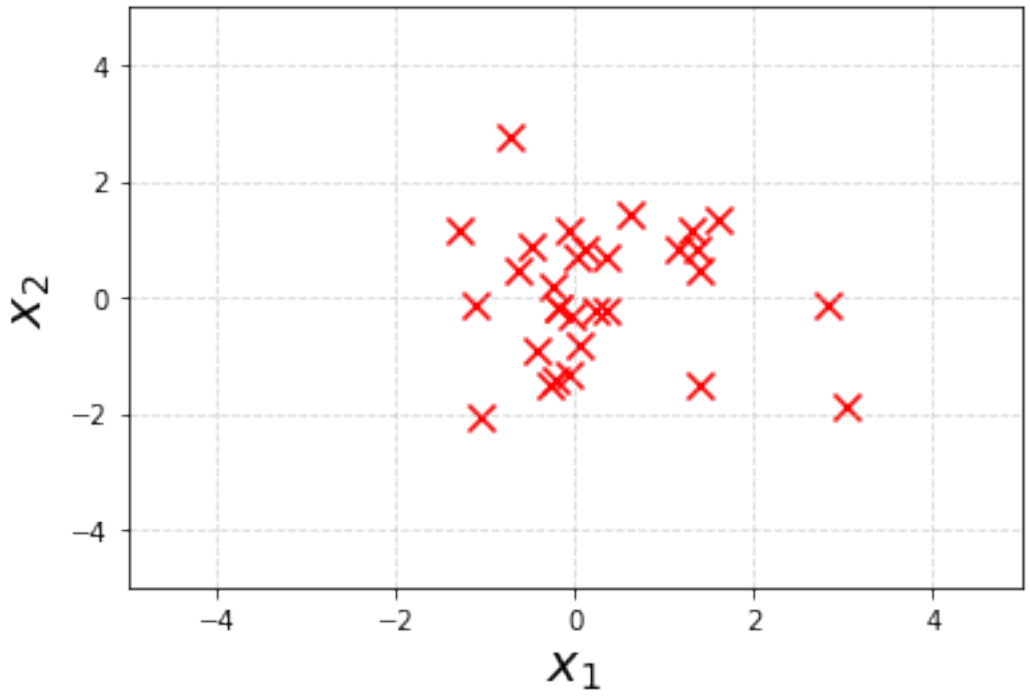

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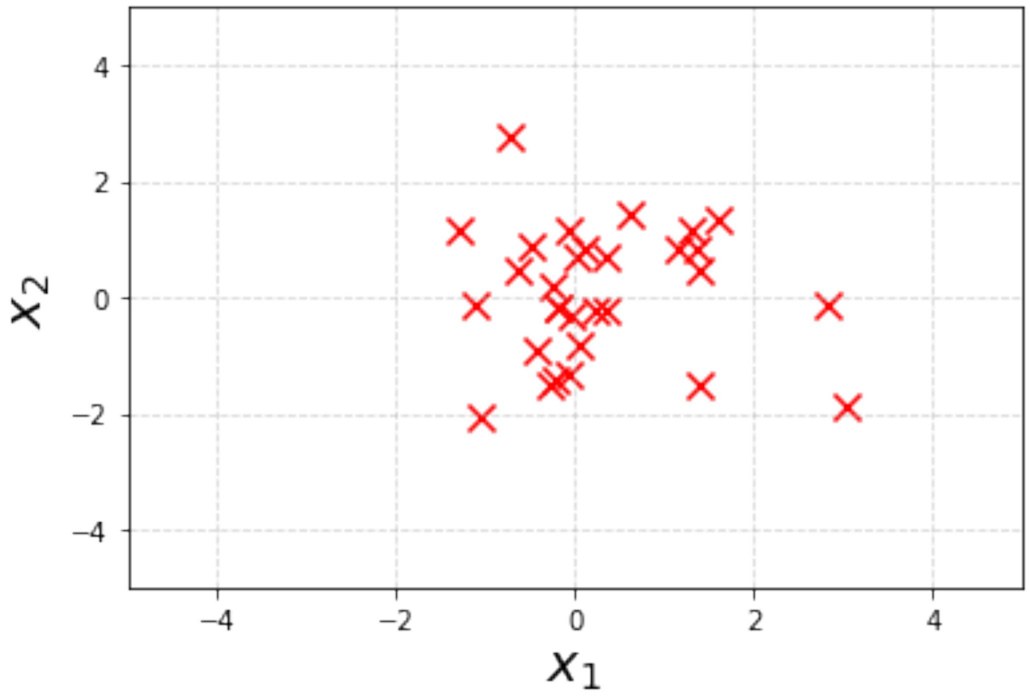

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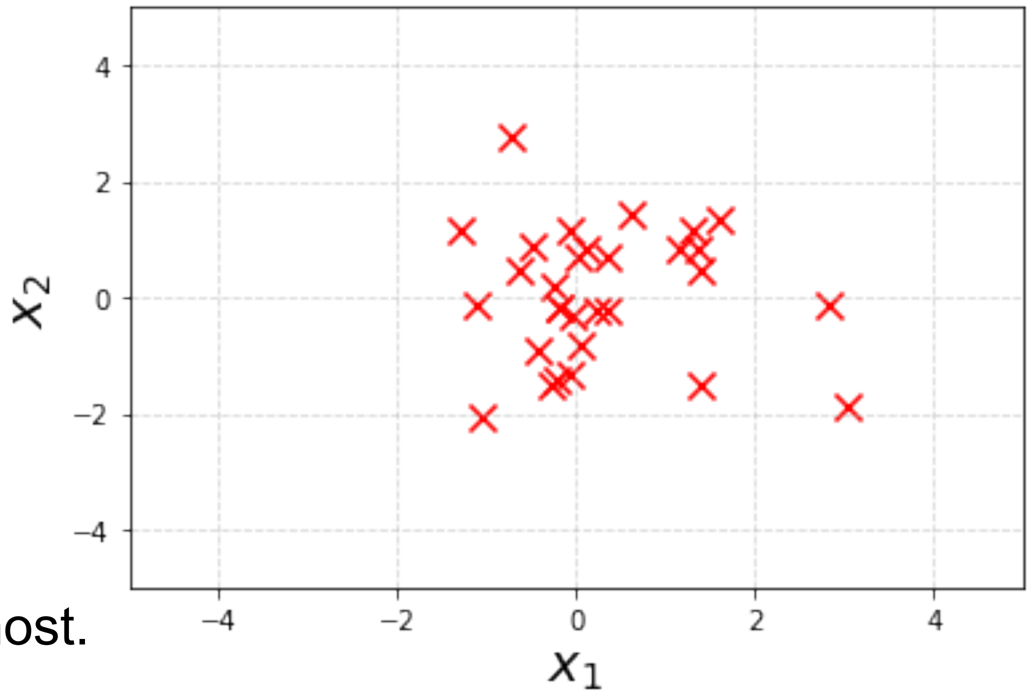
Return parameter value where moments match most.



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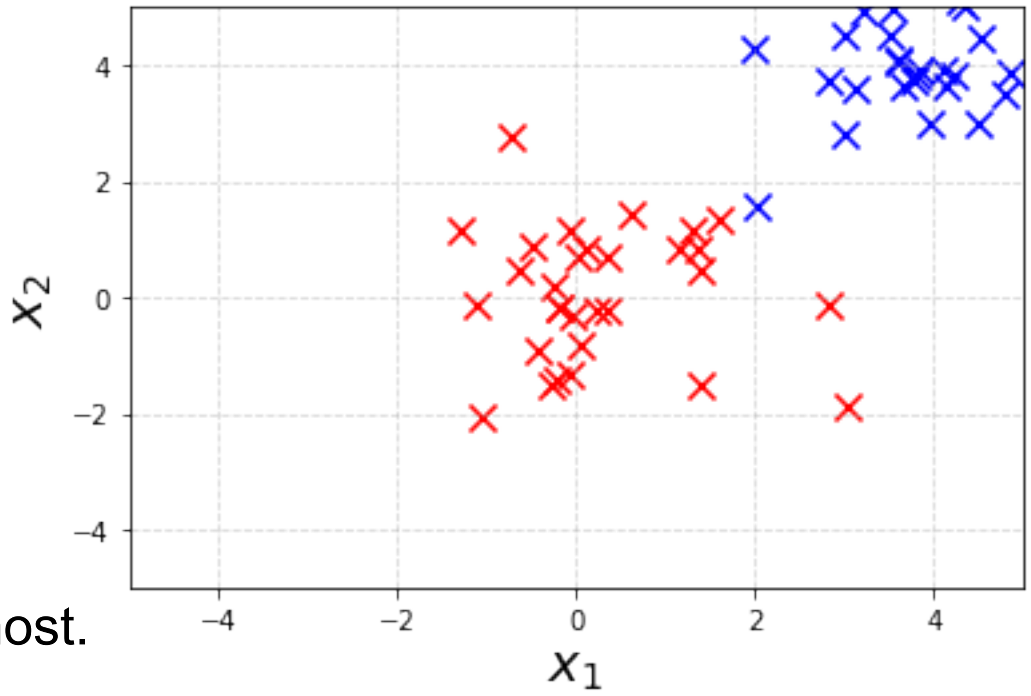

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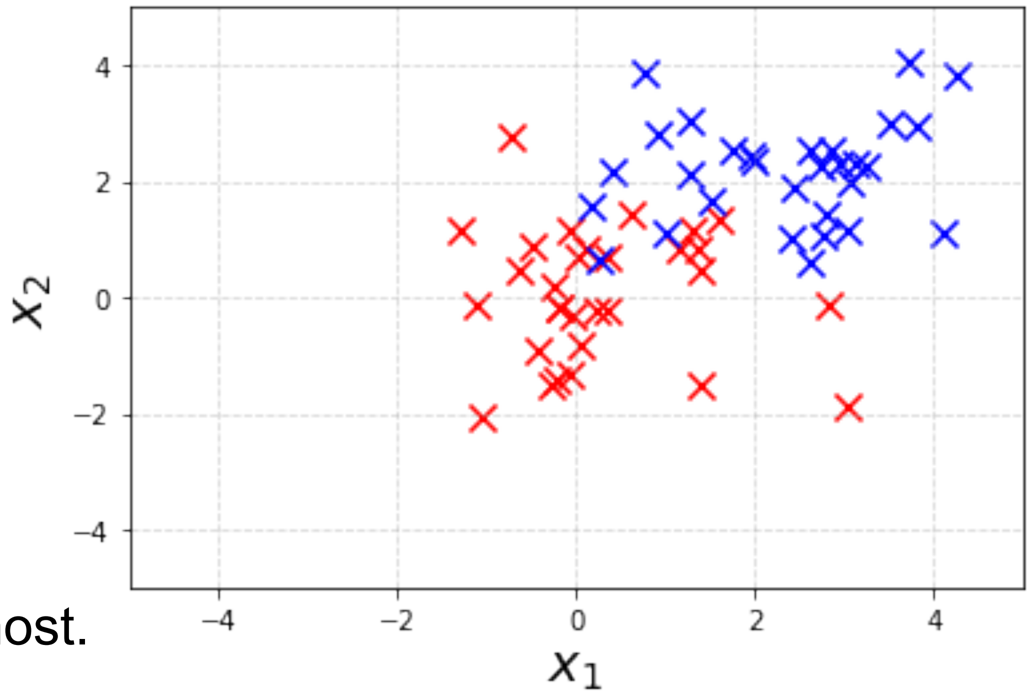

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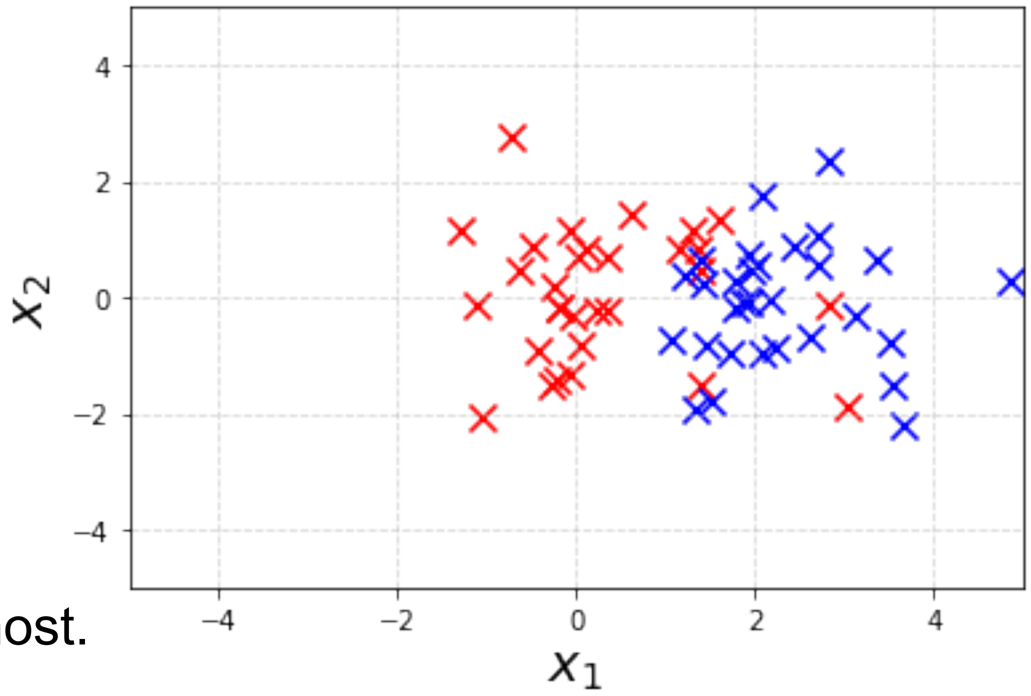

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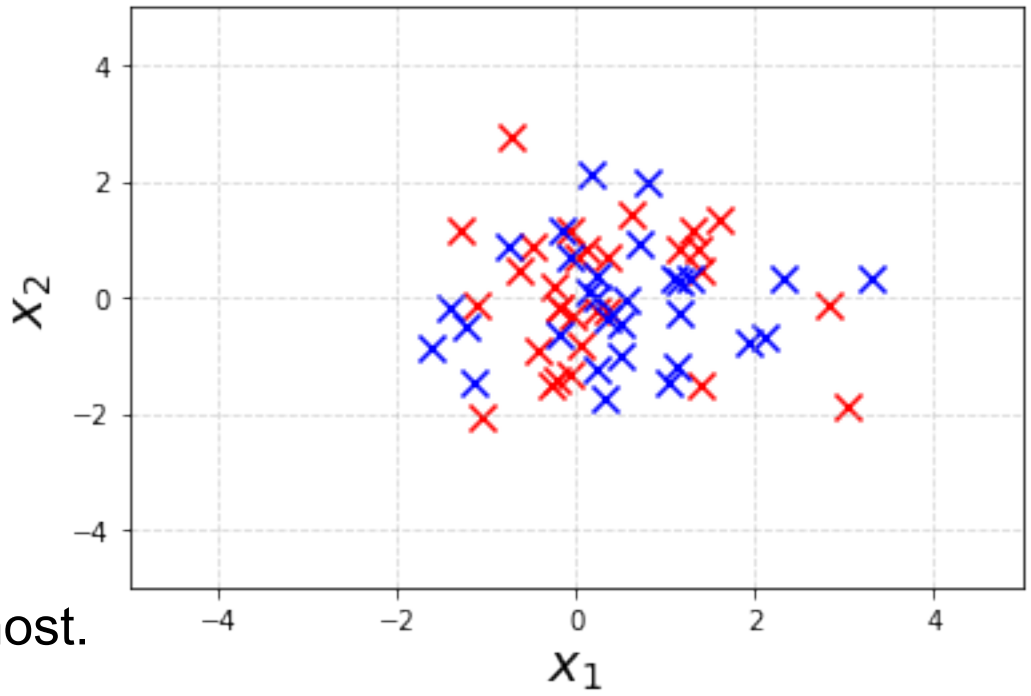

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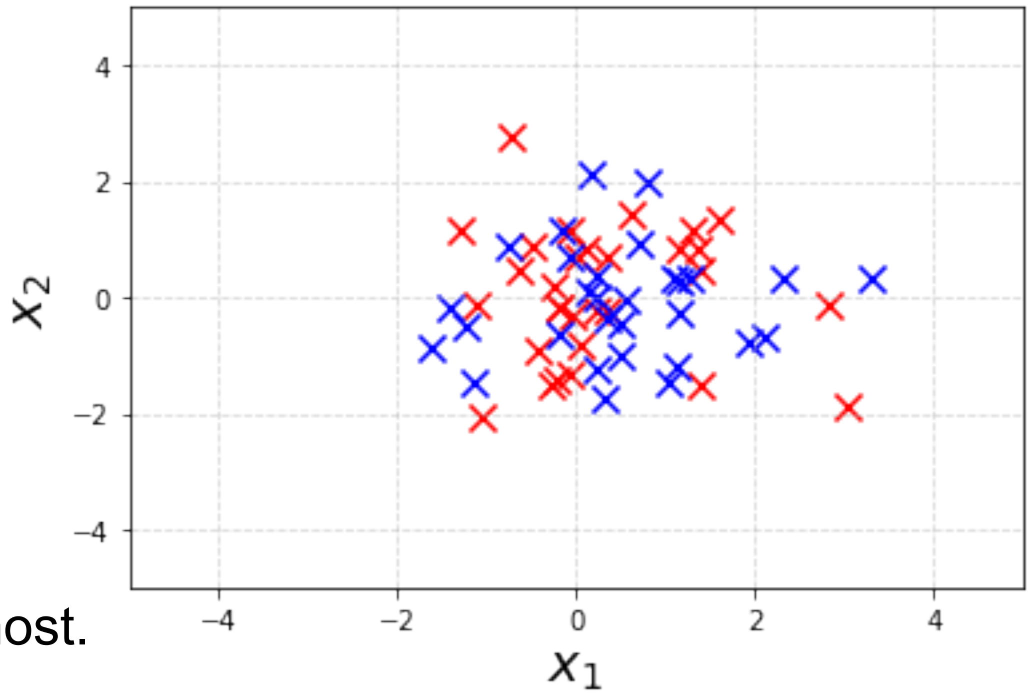

- In practice, this is implemented much more efficiently than by grid search…

Minimum distance estimation for simulators

- A general framework for frequentists: $\theta^* := \arg\min_{\theta \in \Theta} D(\mathbb{P}_\theta, \mathbb{Q})$

Wolfowitz, J. (1957). The minimum distance method. The Annals of Mathematical Statistics, 28(1), 75–88.

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- Can pick our favourite discrepancy/divergence/distance!

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The choice of discrepancy

- Desirable criteria:

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This will typically make things intractable unless $\mathscr{F}$ is picked carefully!

The Wasserstein distance

$$W(\mathbb{P}, \mathbb{Q}) := \sup_{f \in \mathcal{F}_W} \left| \mathbb{E}_{X \sim \mathbb{P}}[f(X)] - \mathbb{E}_{X \sim \mathbb{Q}}[f(X)] \right|$$

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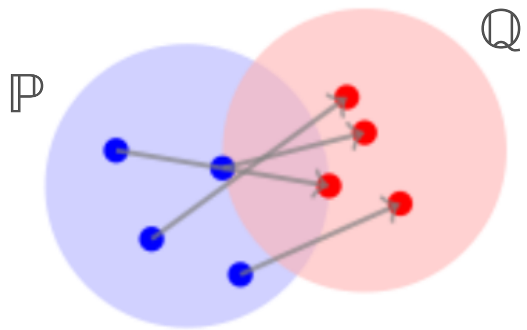

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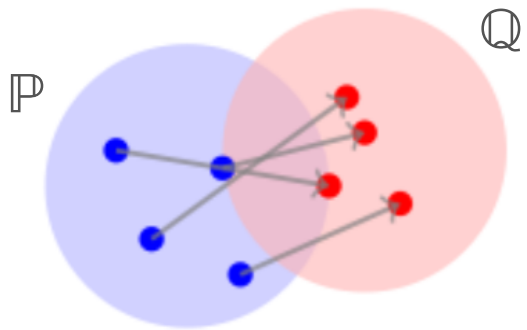

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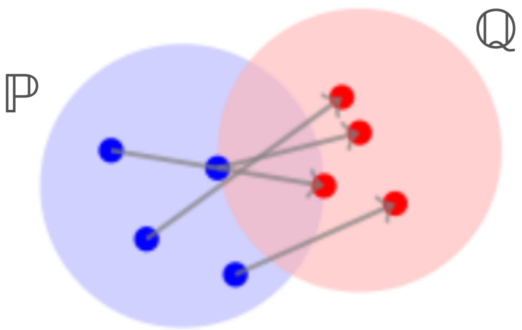

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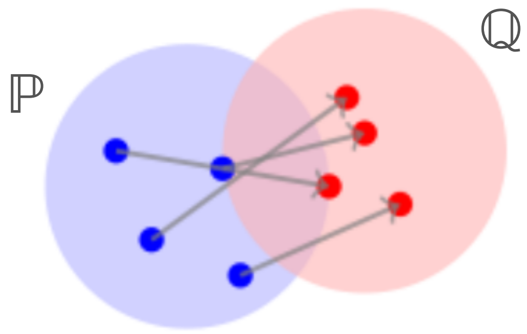

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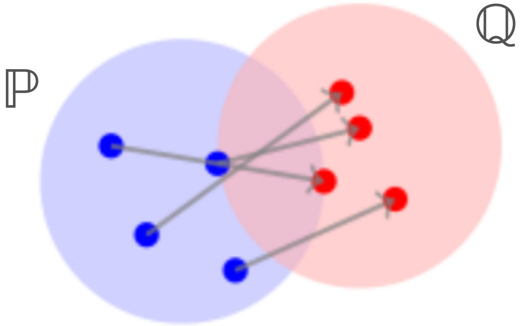

Credit for figure:

Minimum Wasserstein estimators

- Once we have n samples from $\mathbb{P}_\theta$ and $\mathbb{Q}$, this turns into an optimal transport which can be solved in $O(n \log n)$ in $d = 1$ and $O(n^3)$ for $d > 1$.

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- This leads to the following estimator, usually approximated with stochastic optimisation:

$$\hat{\theta}_n := \arg\min_{\theta \in \Theta} \mathsf{W}(\mathbb{P}_\theta, Q_n)$$

Bassetti, F., Bodini, A., & Regazzini, E. (2006). On minimum Kantorovich distance estimators. *Statistics & Probability Letters*, *76*, 1298–1302.

Bernton, E., Jacob, P. E., Gerber, M., & Robert, C. P. (2017). Inference in generative models using the Wasserstein distance. *Information and Inference: A Journal of the IMA*, *8*(4), 657–676.

The maximum mean discrepancy

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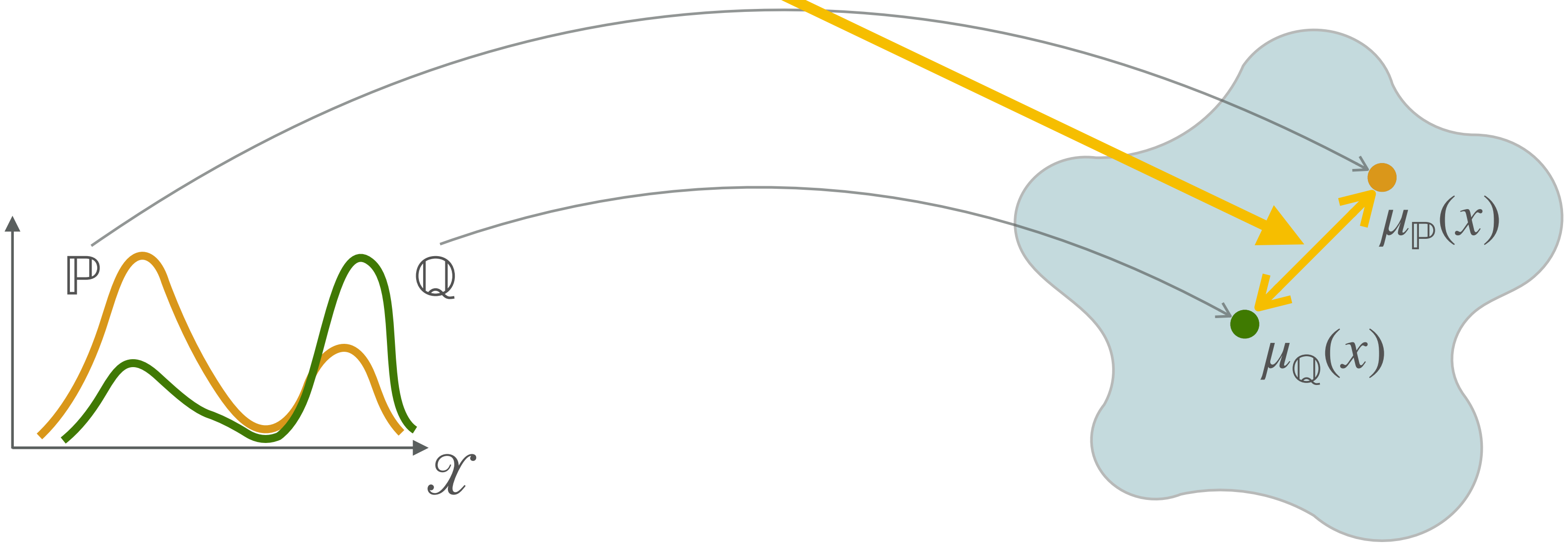

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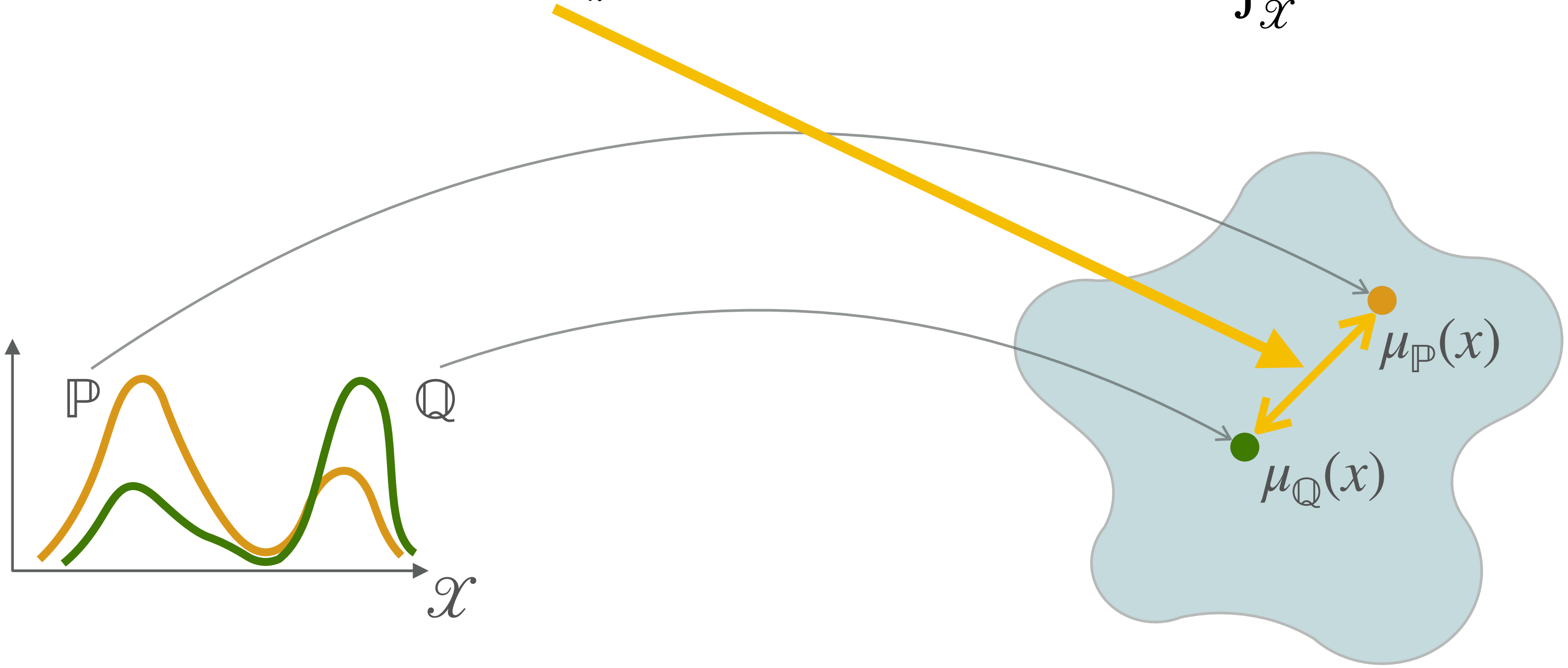

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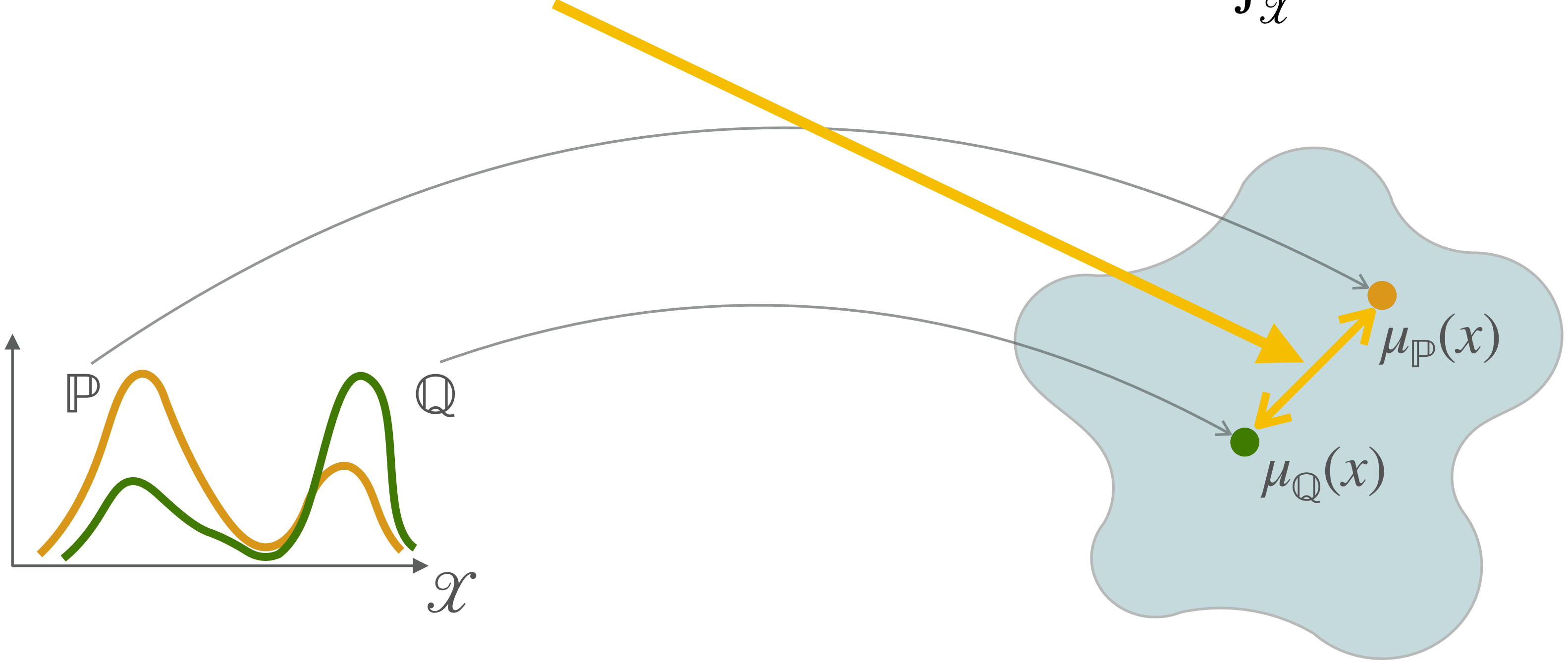

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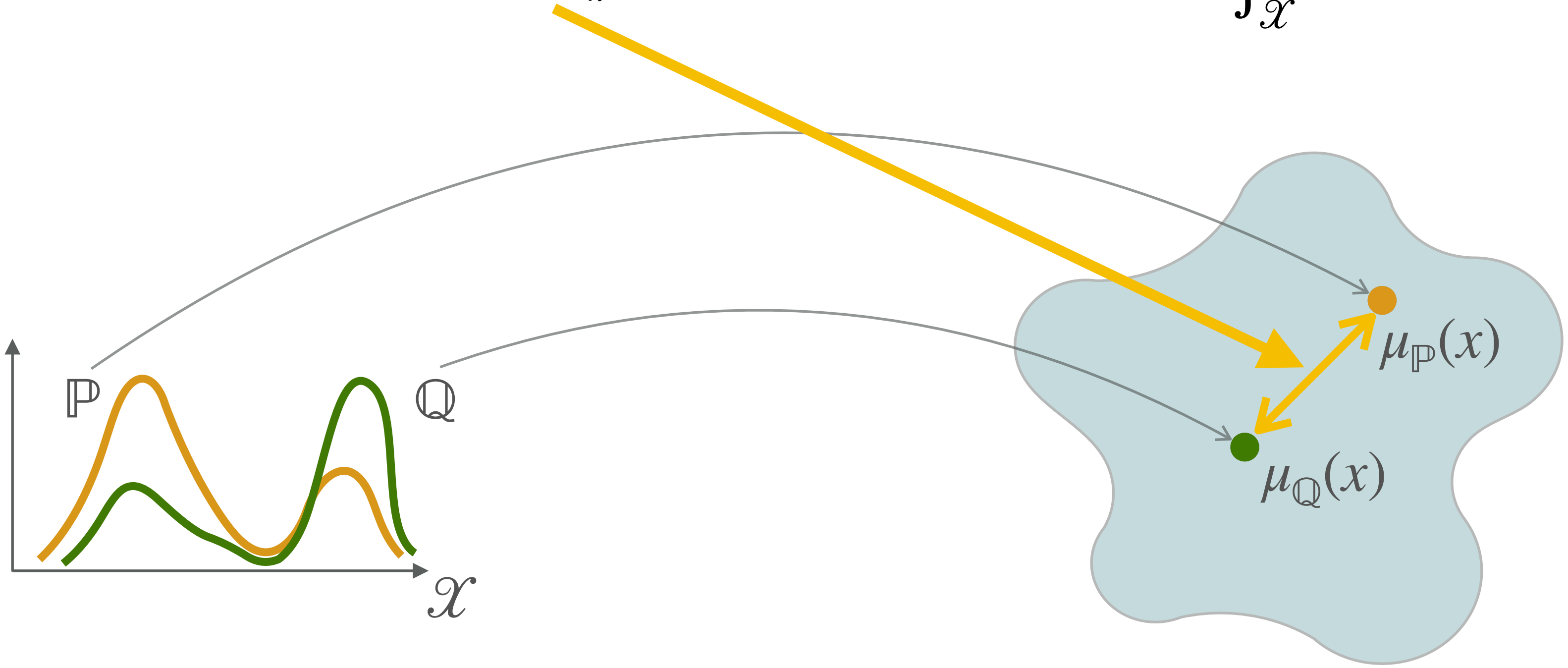

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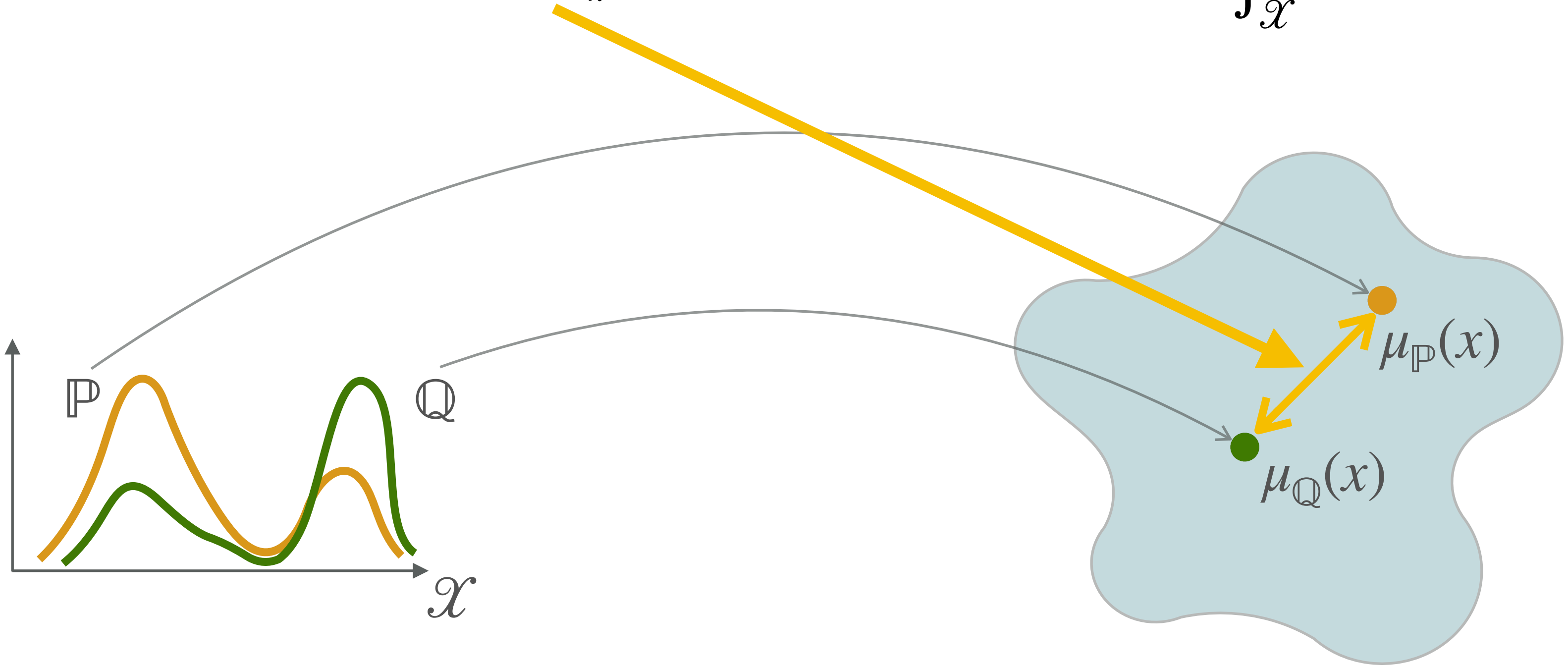

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Minimum MMD estimators

- Thanks to the 'reproducing property', we get:

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Briol, F.-X., Barp, A., Duncan, A. B., & Girolami, M. (2019). Statistical inference for generative models with maximum mean discrepancy. *arXiv:1906.05944*.

Chérief-Abdellatif, B.-E., & Alquier, P. (2022). Finite sample properties of parametric MMD estimation: robustness to misspecification and dependence. *Bernoulli*, *28*(1), 181–213.

Any Questions?

Approximate Bayesian Computation

(From now on we will mostly be Bayesian!)

Approximate Bayesian computation (ABC)

- Recall that we would like to approximate:

$$p(\theta \mid y_1, \ldots, y_n) \propto \prod_{i=1}^{n} p(y_i \mid \theta) p(\theta)$$

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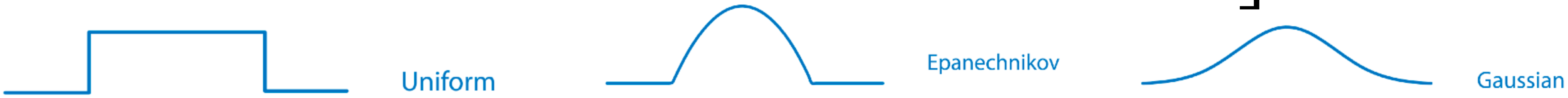

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A simple ABC sampler

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This is still intractable though!!

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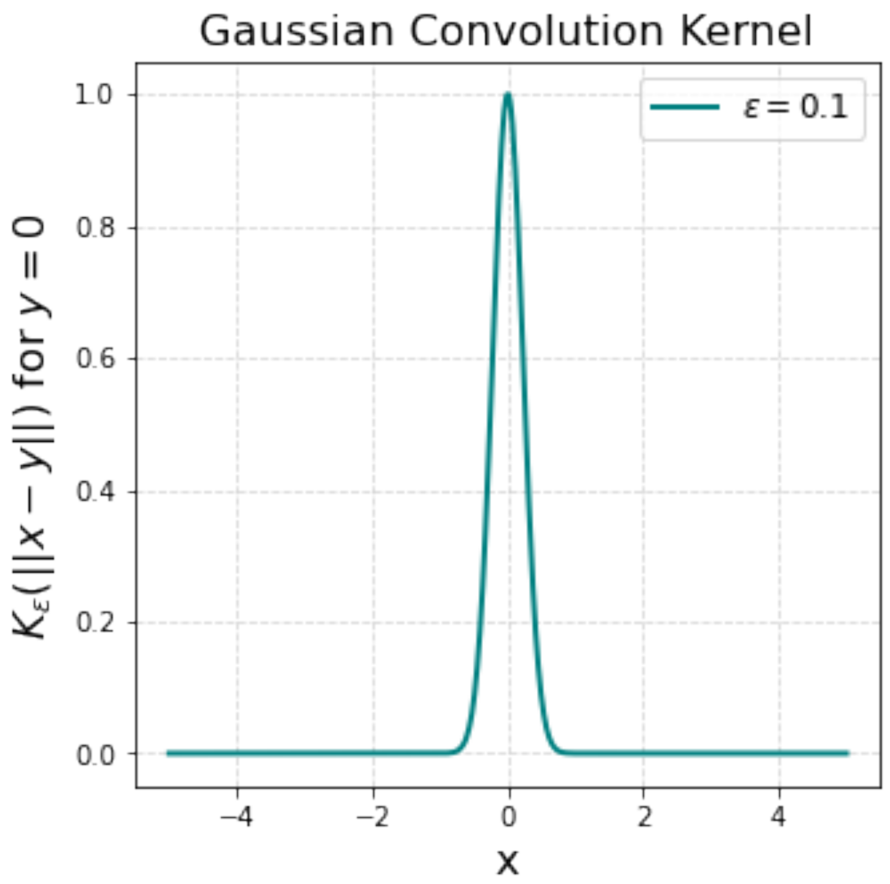
This is just a simple example; there are many more advanced sampling methods (e.g. SMC)

The impact of ϵ



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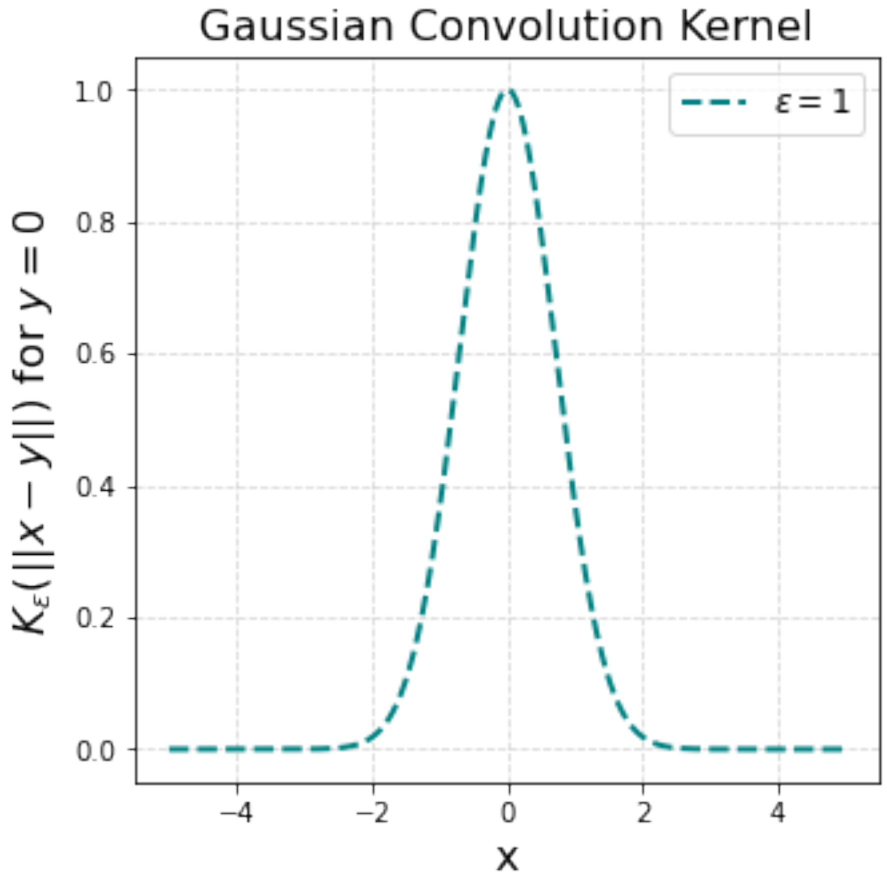

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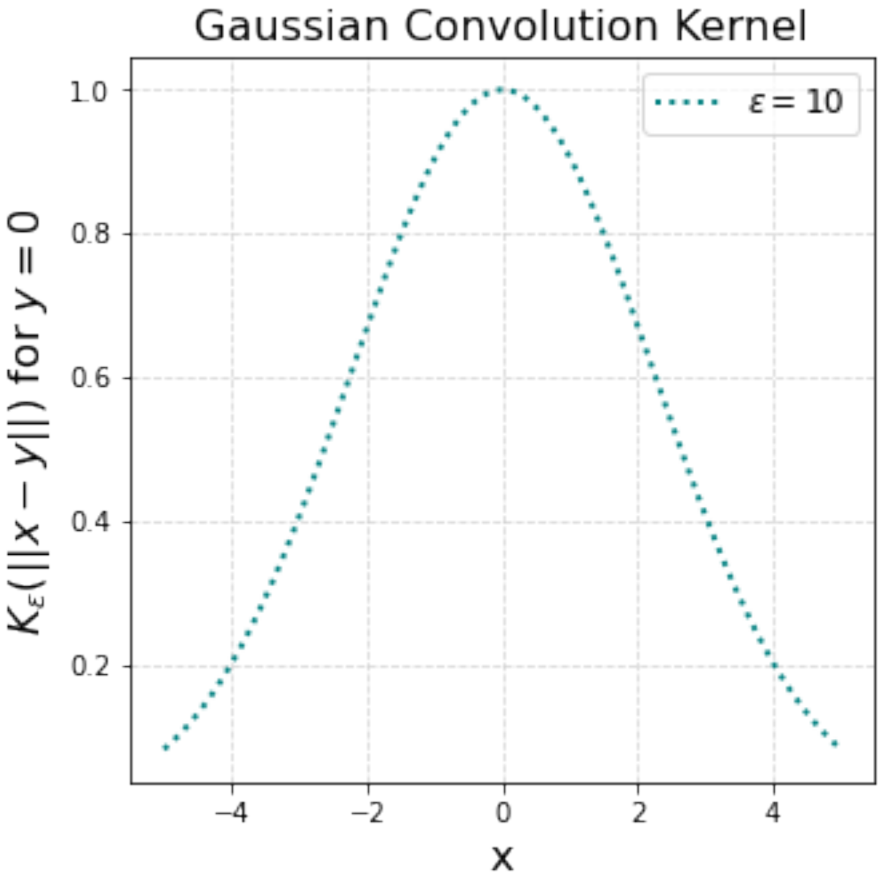

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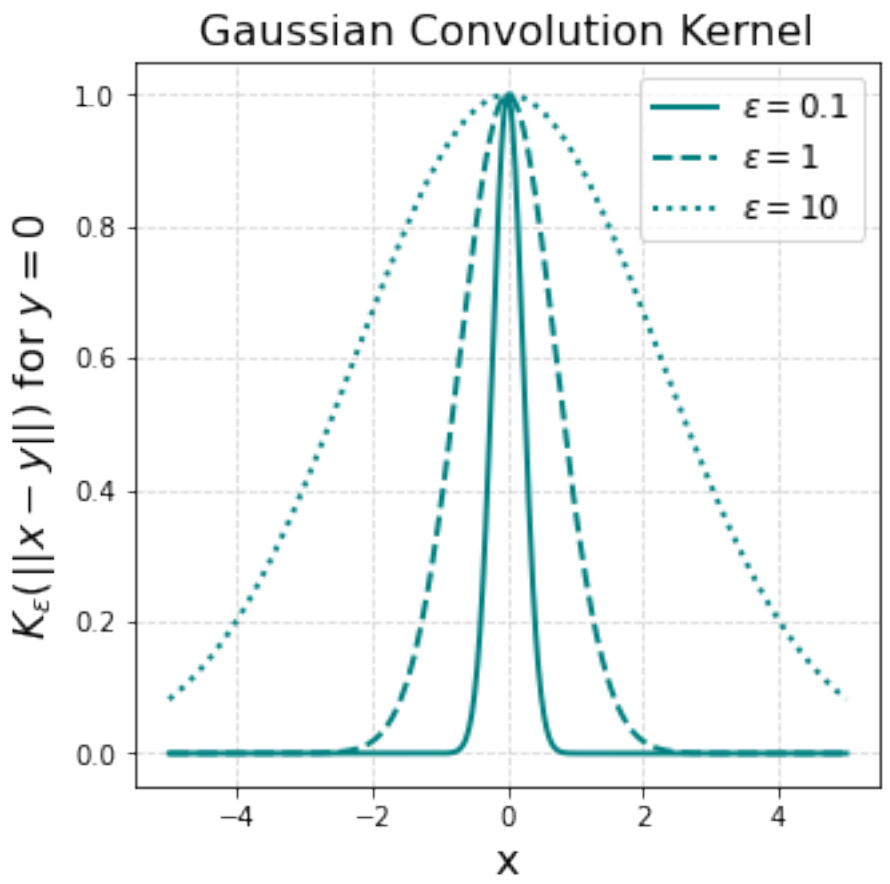

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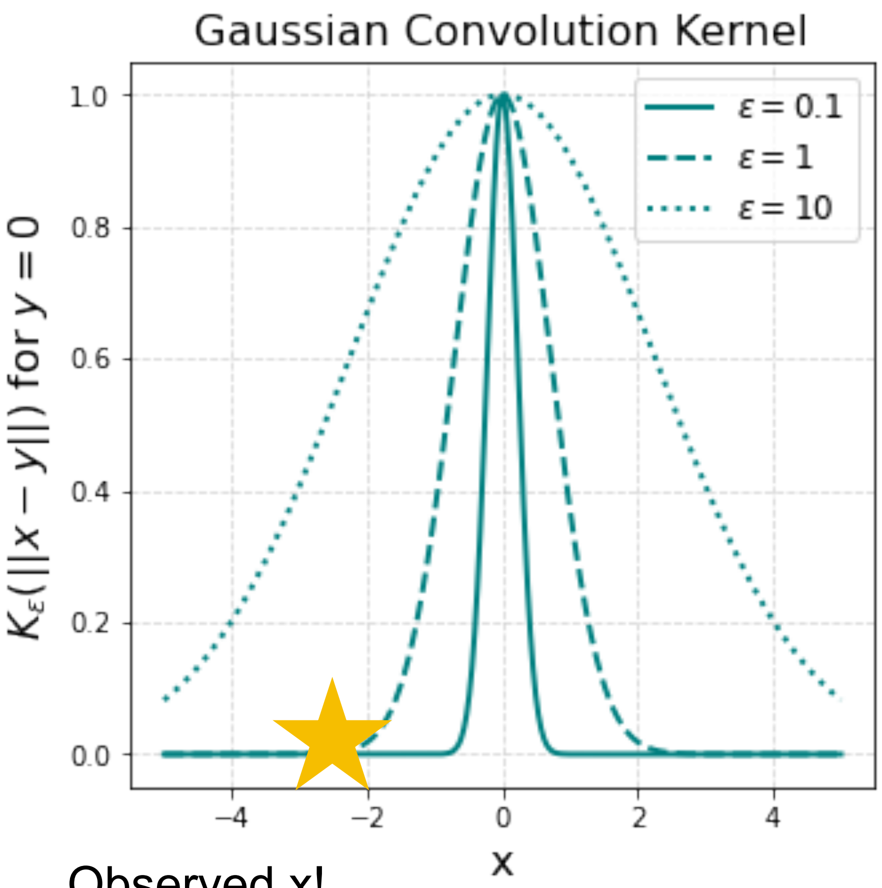
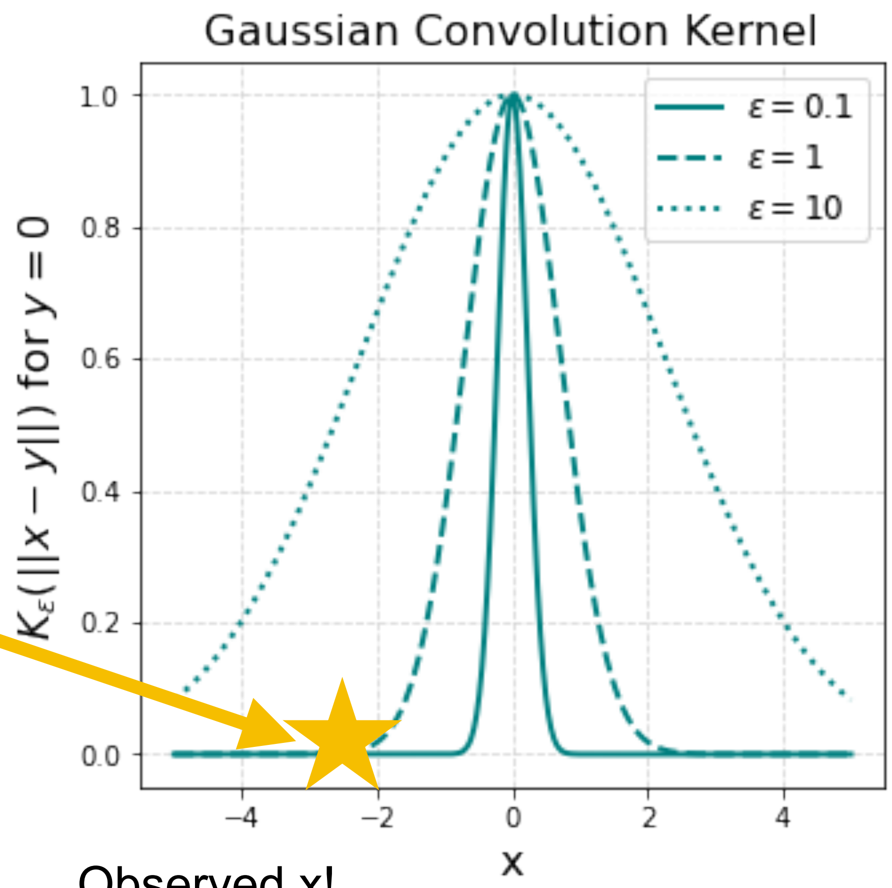


Observed x!

The impact of ϵ

Essentially ignored
when $\epsilon = 0.1$ or $\epsilon = 1$



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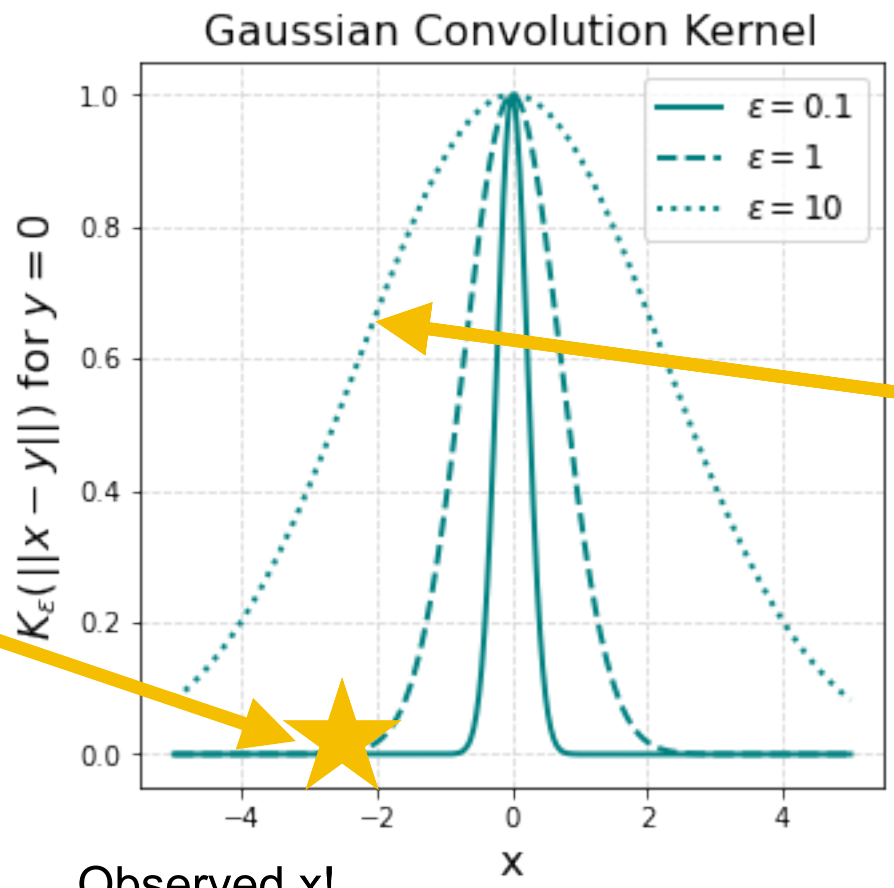

Observed x!

Considered
close if $\epsilon = 10$

Discrepancies-based ABC

$$q_{\mathrm{ABC}}(\theta \mid y_1, \ldots, y_n) \propto \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} K_\epsilon(\cancel{\|x - y\|}) \prod_{i=1}^{n} p(x_i \mid \theta) p(\theta) dx_1 \ldots dx_n$$

$$K_\epsilon\left(D\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}, \frac{1}{n}\sum_{i=1}^{n}\delta_{y_i}\right)\right) = K_\epsilon\left(D\left((\mathbb{P}_\theta)_n, \mathbb{Q}_n\right)\right)$$

Park, M., Jitkrittum, W., & Sejdinovic, D. (2016). K2-ABC: Approximate bayesian computation with kernel embeddings. *AISTATS*, *51*, 398–407.

Bernton, E., Jacob, P. E., Gerber, M., & Robert, C. P. (2019). Approximate Bayesian computation with the Wasserstein distance. *JRSSB*, *81*(2), 235–269.

Legramanti, S., Durante, D., & Alquier, P. (2025). Concentration and robustness of discrepancy-based ABC via Rademacher complexity. *The Annals of Statistics*, *53*(1), 37–60.

Any Questions?

ML approaches to SBI

We have now already covered the state-of-the-art until 2020-ish!

SBI with conditional density estimators

- I probably don't need to convince you that machine learning methods are very good at emulation…. How can we use this for Bayes?

$$p(\theta \mid y_1, \ldots, y_n) \propto \prod_{i=1}^{n} p(y_i \mid \theta) p(\theta)$$

Zammit-mangion, A., Sainsbury-Dale, M., & Huser, R. (2025). Neural methods for amortized parameter inference. *Annual Review of Statistics and Its Application*, *12*, 311–335.

Deistler, M., Boelts, J., Steinbach, P., Moss, G., Moreau, T., Gloeckler, M., Rodrigues, P. L. C., Linhart, J., Lappalainen, J. K., Miller, B. K., Gonçalves, P. J., Lueckmann, J.-M., Schröder, C., & Macke, J. H. (2025). Simulation-based inference: A practical guide. *arXiv:2508.12939*.

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- We will start by emulating the likelihood; i.e. we want a flexible class: $\{q_\phi(x \mid \theta)\}_{\Phi \in \Phi}$

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- We can increase the flexibility:

$$q_\phi(x \mid \theta) = \sum_{c=1}^{C} w_c(\phi; \theta) \mathcal{N}(x \mid \mu_c(\phi; \theta), \Sigma_c(\phi; \theta))$$

Bishop, C. M. (1994). *Mixture density networks* (NCRG/94/004).

Transformations and densities

- Consider some base distribution p_v and some transformation T such that

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- How do we design T if we want the density model to be very flexible?

→ Use neural networks!!

Normalising flows (I)

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We can also parametrise them!

Normalising flows (I)

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- We end up with a normalising flow:

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Normalising flows (I)

- Note that we can compose such maps and keep their desirable properties:

$$T_{\phi,\theta} = T^K_{\phi,\theta} \circ \ldots \circ T^2_{\phi,\theta} \circ T^1_{\phi,\theta}$$

Straightforward to create conditional density!

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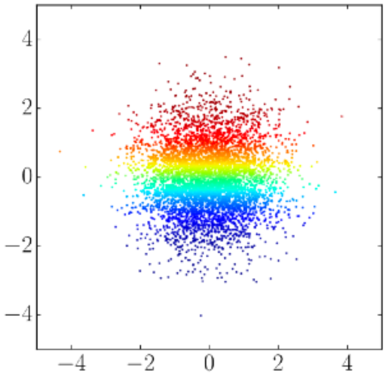
- Terminology: Are normalising flows simulators?

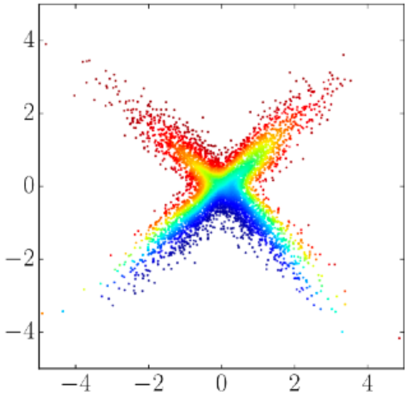
→ They can be, but (similarly to diffusion models) they do not typically encode any science, they are just constructed to be very flexible models!

Normalising flows (III)

$p_v(v)$



$q_\phi(x)$



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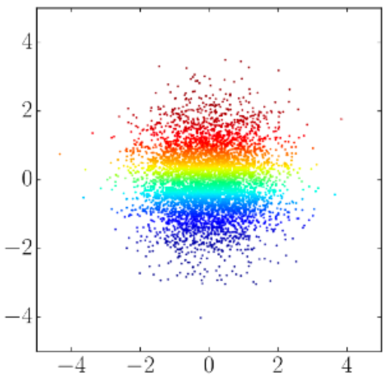
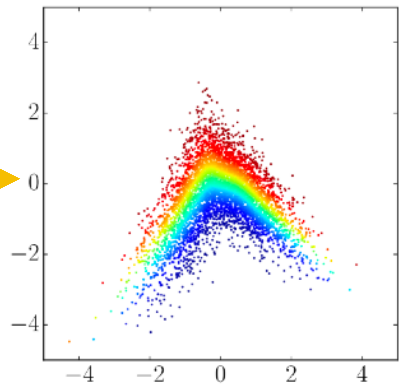


$\xrightarrow{T_\phi^1}$



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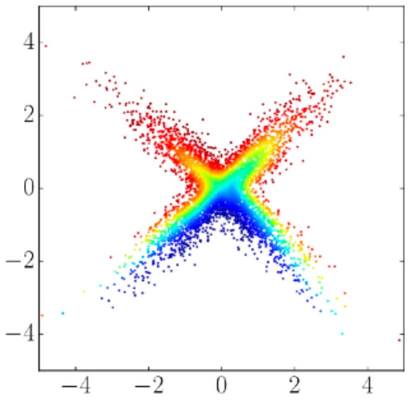

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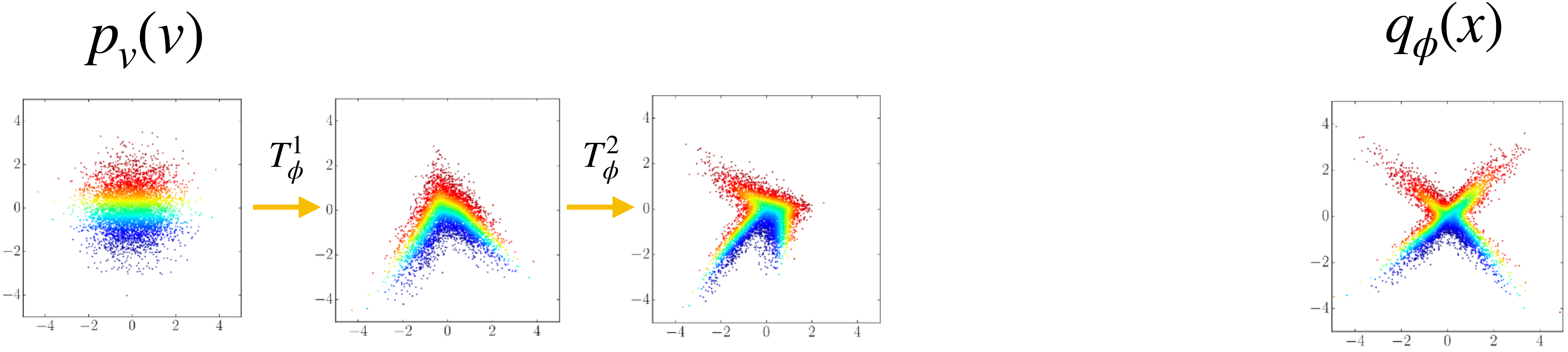

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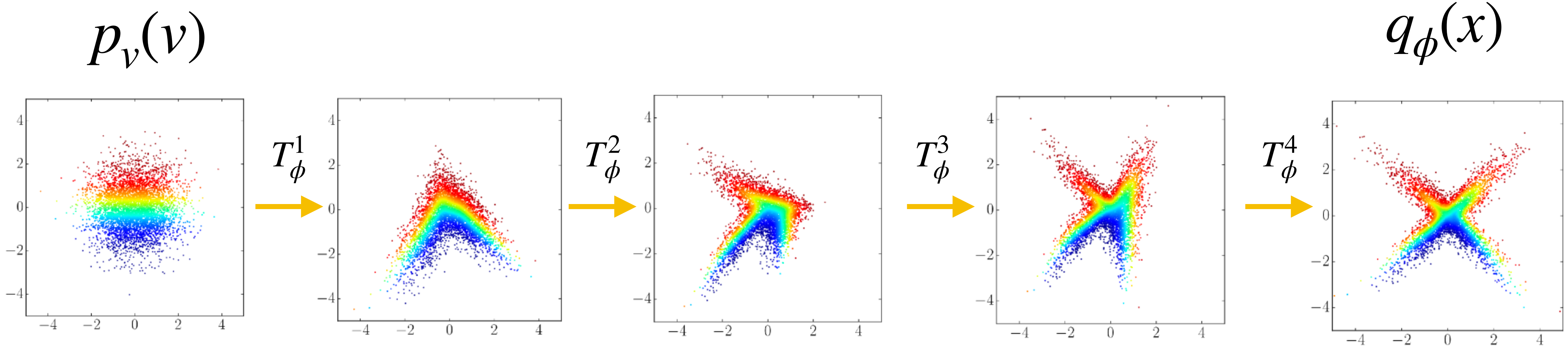

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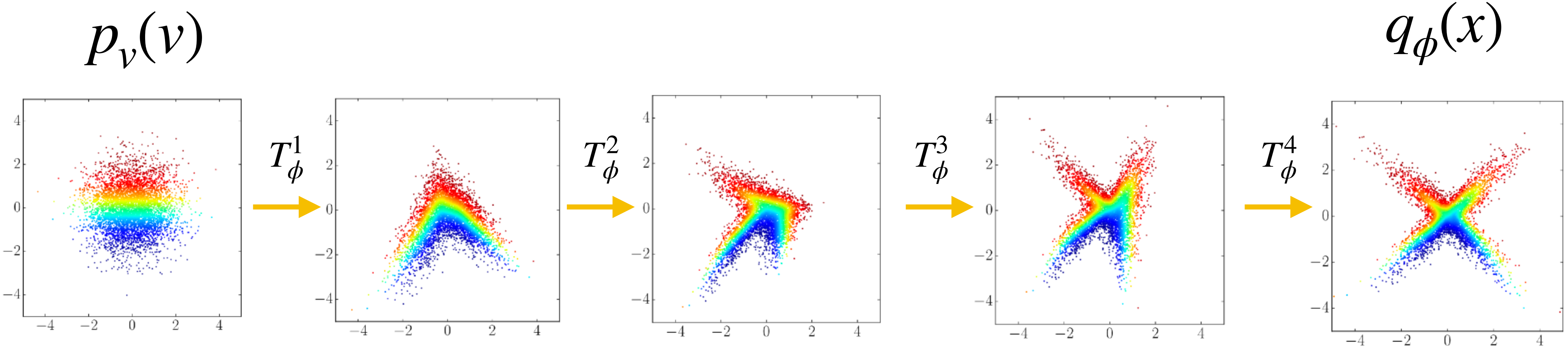

The composition of relatively simple transformations can give fairly complex maps!

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Neural likelihood estimation (NLE)

- **Step 1:** train $q_\phi(x \mid \theta)$ to approximate the likelihood using samples from the prior ($\theta_1, \ldots, \theta_n \sim p(\theta)$) and simulator ($x_i \sim p(\cdot \mid \theta_i)$):

$$\hat{\phi}_n := \arg\min_{\phi \in \Phi} \ell_{\mathrm{NLE}}(\phi), \quad \ell_{\mathrm{NLE}}(\phi) = -\frac{1}{n}\sum_{i=1}^{n} \log q_\phi(x_i \mid \theta_i) \approx -\mathbb{E}_{\theta\sim p(\theta)}[\mathbb{E}_{x\sim p(\cdot\mid\theta)}[\log q_\phi(x \mid \theta)]]$$

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- **Step 2:** Approximate posterior (MCMC, VI) constructed with surrogate likelihood!

$$p_{\text{NLE}}(\theta \mid y_1, \ldots, y_n) \propto \prod_{i=1}^{n} q_{\hat{\phi}_n}(y_i \mid \theta) p(\theta)$$

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Amortisation for NLE

- Recall the NLE posterior:

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- What if we get some new observations $\tilde{y}_1, \ldots, \tilde{y}_n$?

→ We already have an emulator of the likelihood, so we just need to use it!

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We still need to re-run MCMC/VI though… We are **partially amortised**.

Neural posterior estimation (NPE)

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Papamakarios, G., & Murray, I. (2016). Fast e-free inference of simulation models with Bayesian conditional density estimation. *NeurIPS*, 1036–1044.

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Neural posterior estimation (NPE)

- **Step 1:** train $q_\phi(\theta \mid x)$ to approximate the posterior using samples from the prior ($\theta_1, \ldots, \theta_n \sim p(\theta)$) and simulator ($x_i \sim p(\cdot \mid \theta_i)$):

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- **Step 2:** Condition on the observed data:

$$p_{\mathrm{NPE}}(\theta \mid y_1, \ldots, y_n) = q_{\hat{\phi}_n}(\theta \mid y_1, \ldots, y_n)$$

Papamakarios, G., & Murray, I. (2016). Fast e-free inference of simulation models with Bayesian conditional density estimation. *NeurIPS*, 1036–1044.

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Amortisation of NPE

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$$p_{\text{NPE}}(\theta \mid \tilde{y}_1, \ldots, \tilde{y}_n) = q_{\hat{\phi}_n}(\theta \mid \tilde{y}_1, \ldots, \tilde{y}_n)$$

- We have a direct handle on the new posterior; no need for MCMC/VI!

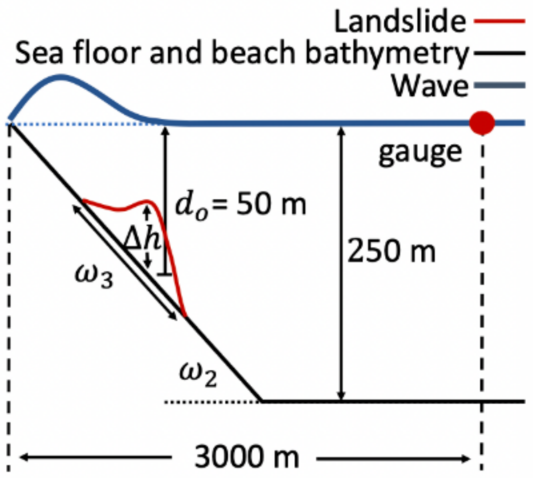
→ We are **fully amortised**.

Any Questions?

Challenges with existing SBI methods

Challenge 1: Expensive simulators

<u>Example 1:</u>



≈ 2 hours per sim on laptop

Li, K., Giles, D., Karvonen, T., Guillas, S., & **Briol, F.-X**. (2023). Multilevel Bayesian quadrature. *AISTATS*, 1845–1868.

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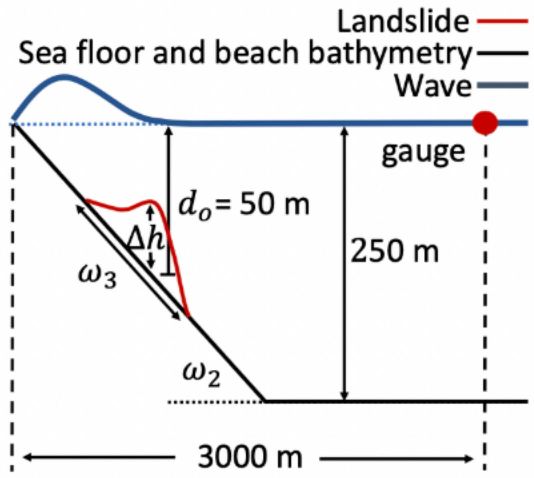

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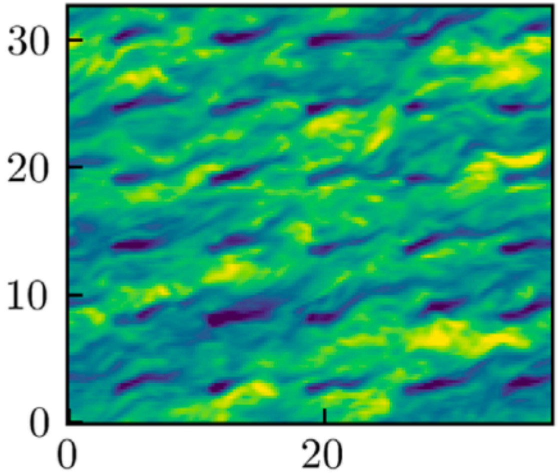
Example 2:



≈ 100 hours per sim on Met Office cluster

Kirby, A., **Briol, F.-X.**, Dunstan, T. D., & Nishino, T. (2023). Data-driven modelling of turbine wake interactions and flow resistance in large wind farms. *Wind Energy*, *26*(9), 875–1011.

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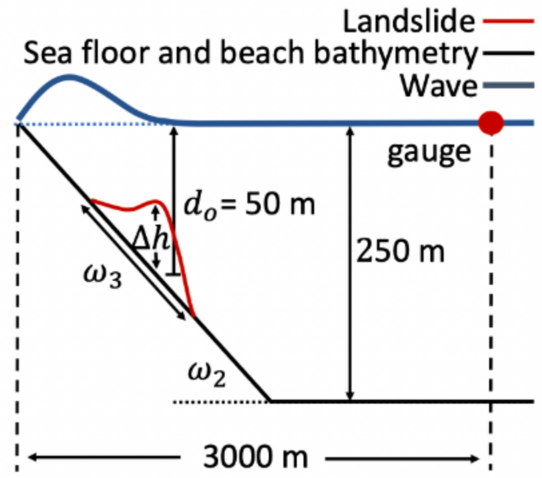

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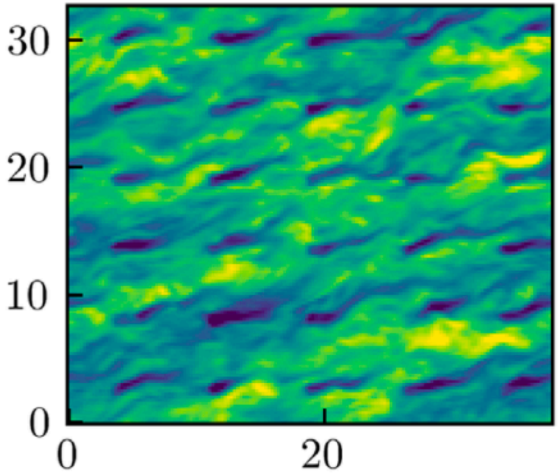

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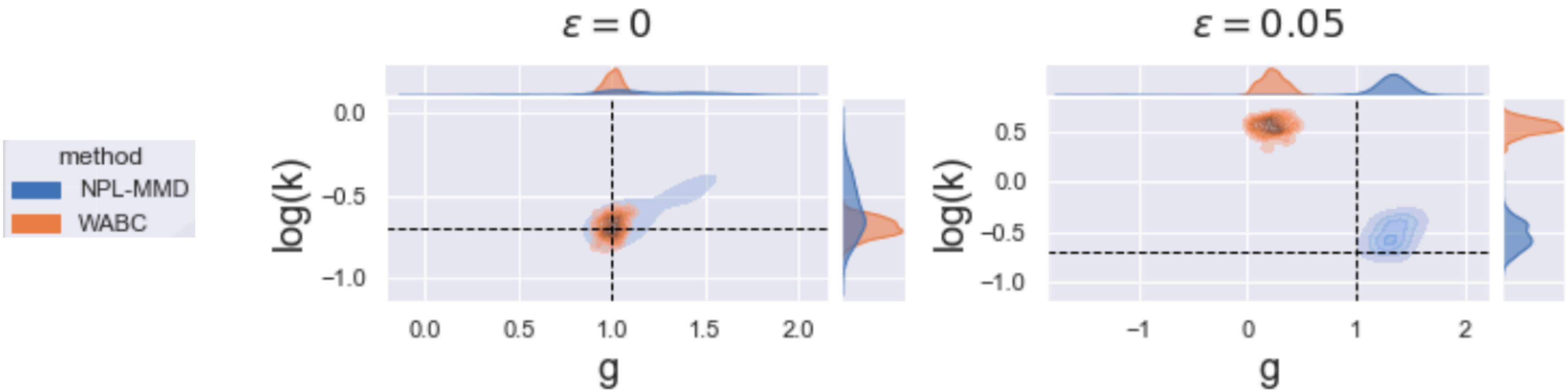
Currently out of reach of modern SBI methods!

Challenge 2: Model misspecification



G-and-k distribution

Dellaporta, C., Knoblauch, J., Damoulas, T. & **Briol, F-X** (2022). Robust Bayesian inference for simulator-based models via the MMD posterior bootstrap. AISTATS, 943-970. Best paper award.

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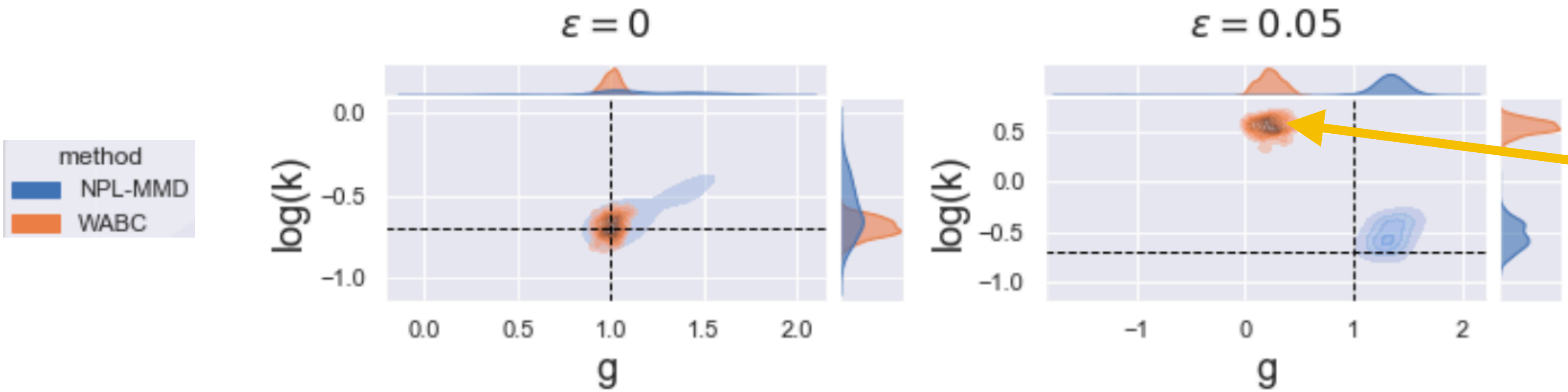

A tiny % of corrupted observations is enough to seriously affect the posterior.

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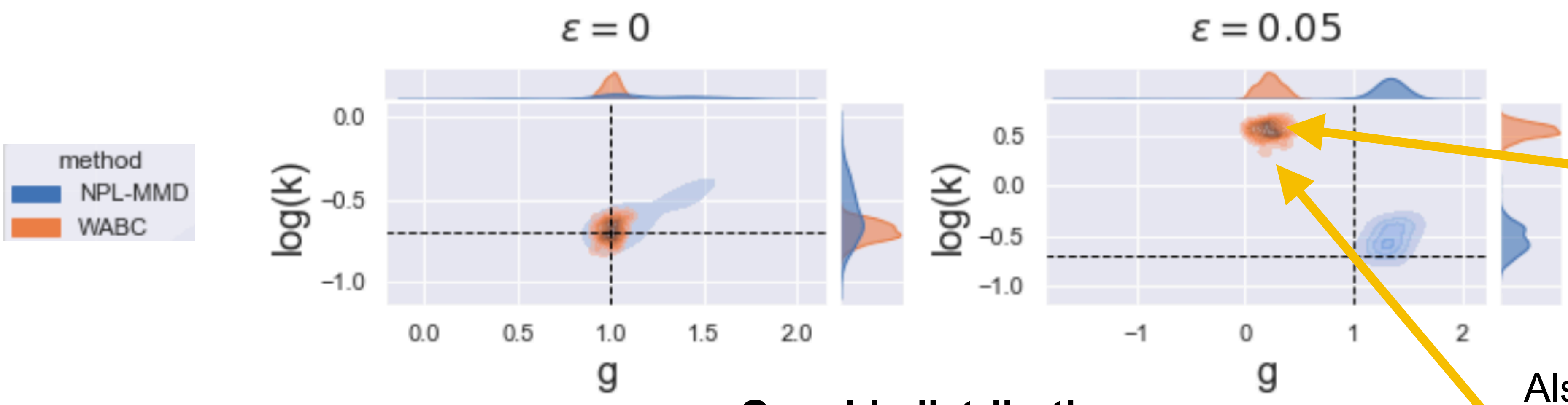

A tiny % of corrupted observations is enough to seriously affect the posterior.

Also leads to serious overconfidence!

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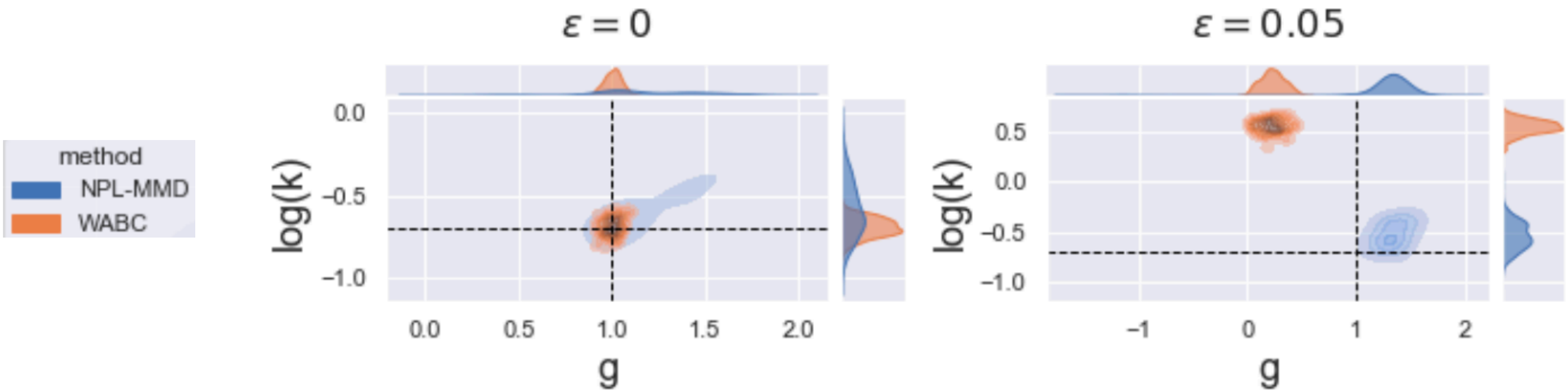

G-and-k distribution

Currently very few robust methods with theoretical guarantees

Challenge 3: Over-confidence

Published in Transactions on Machine Learning Research (11/2022)

A Trust Crisis In Simulation-Based Inference? Your Posterior Approximations Can Be Unfaithful

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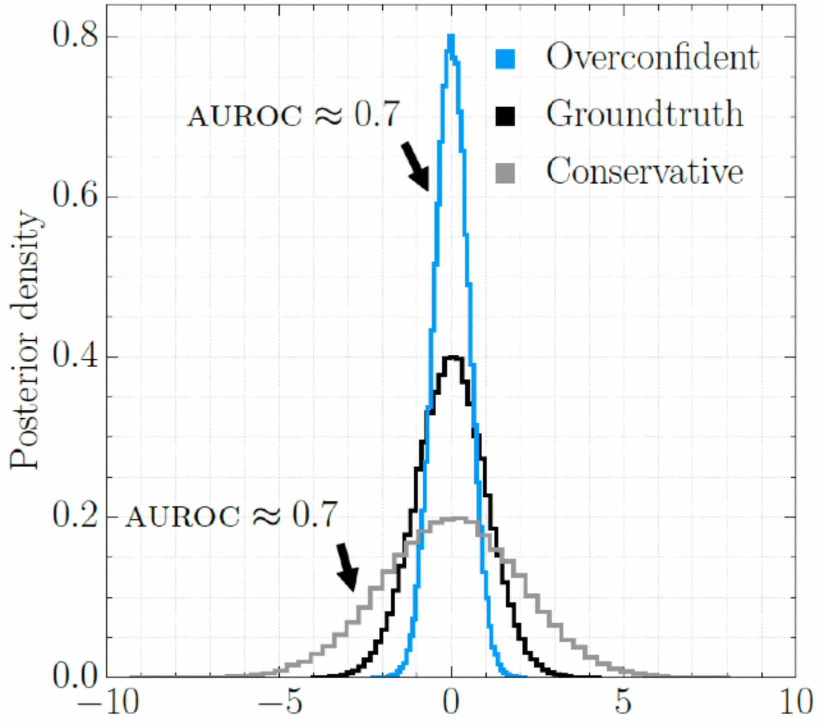

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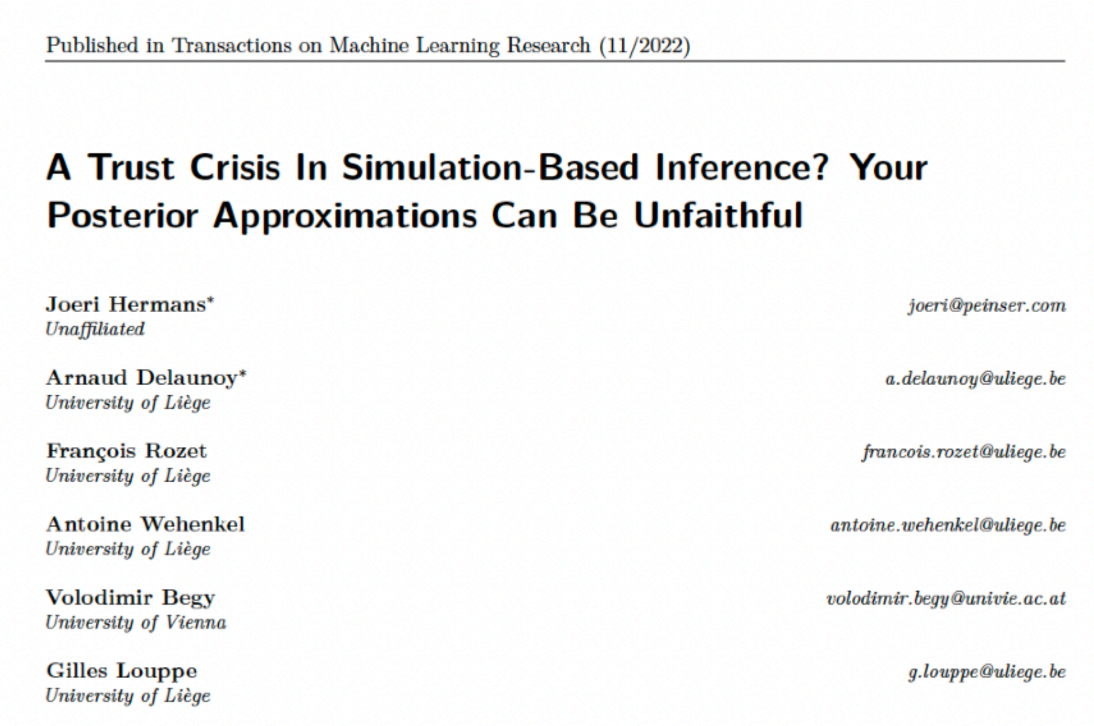
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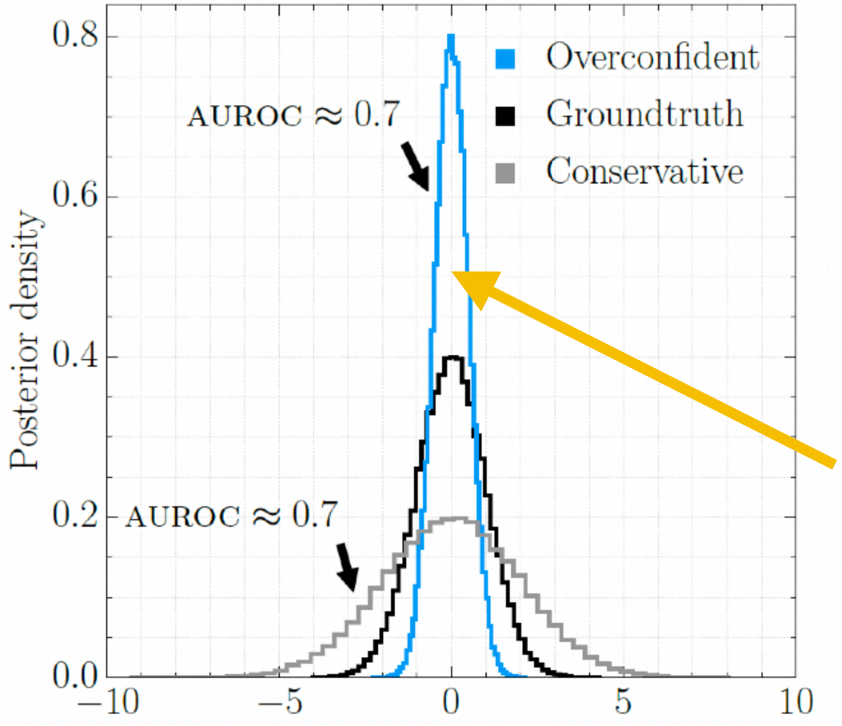

Overconfident =
Too narrow!

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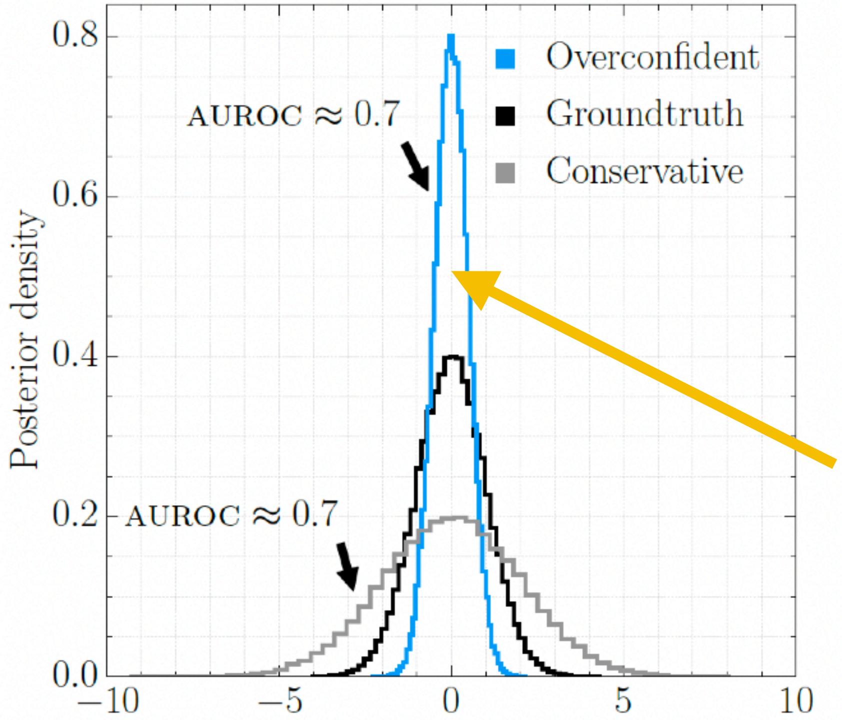

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Observation 1 All benchmarked algorithms may produce non-conservative posterior approximations.

Challenge 4: High-dimensionality

- As with everything in stats/ML, the curse of dimensionality hurts us…. Computing distances or estimating densities is very tough!

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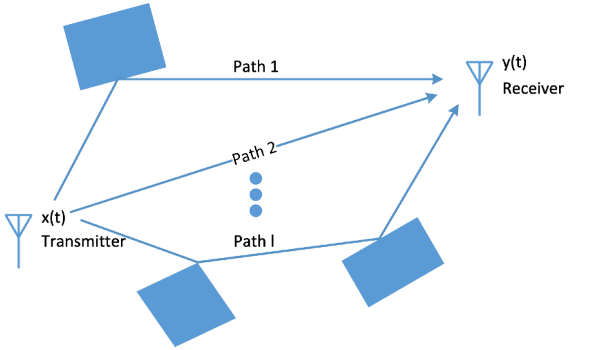

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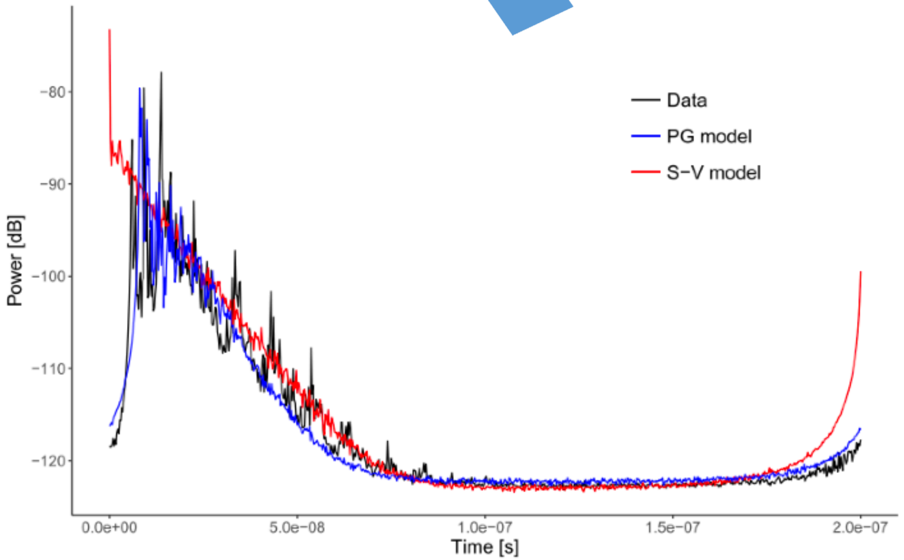
- Remember the radio-propagation example. The dimension is typically around 800….



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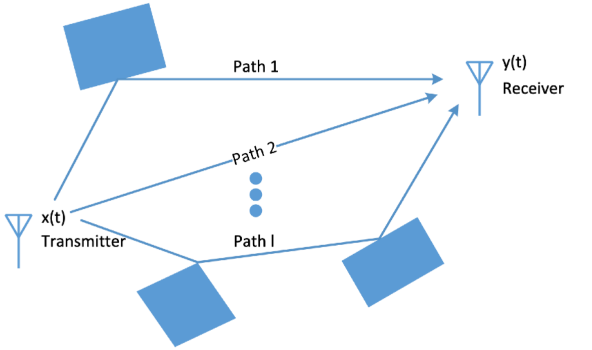

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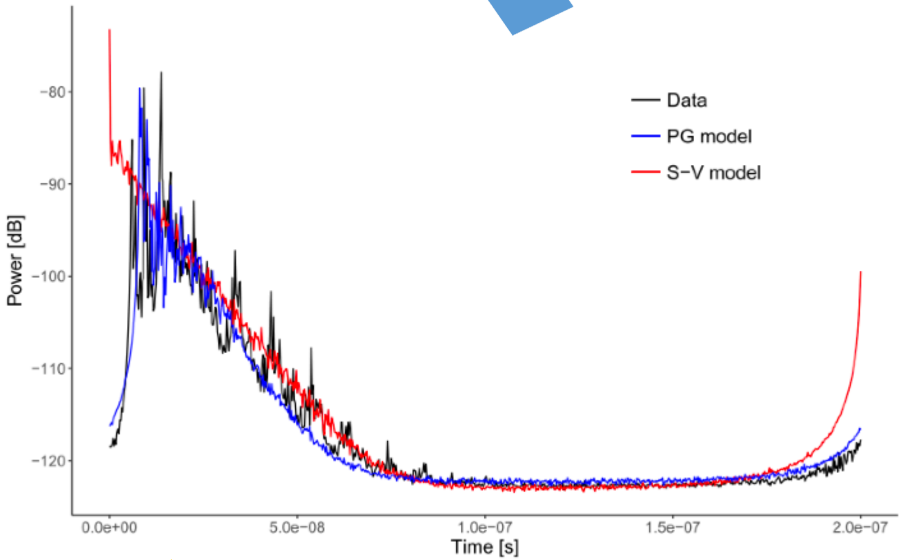

Turns out 4 dimensional summary is practically sufficient here!

Bharti, A., **Briol, F.-X.**, & Pedersen, T. (2021). A general method for calibrating stochastic radio channel models with kernels. *IEEE Transactions on Antennas and Propagation*, *70*(6), 3986–4001.

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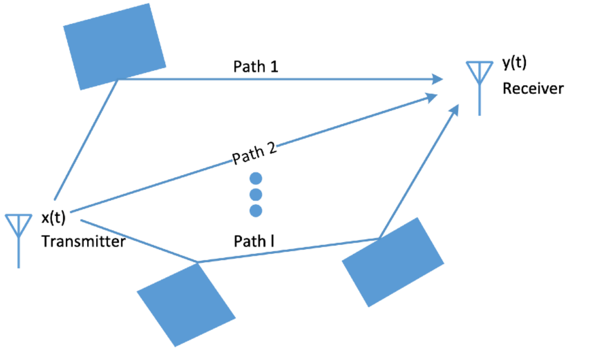

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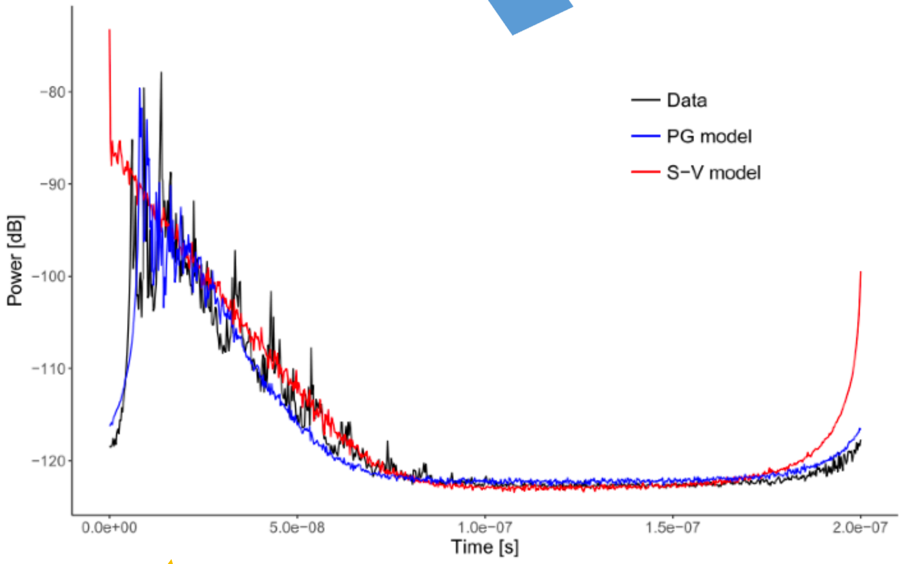
- We therefore end up working with **summary statistics,** either hand-crafted or learnt via a neural network (i.e. a 'summary network').



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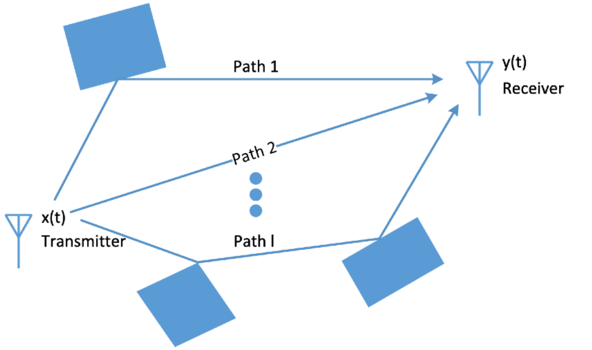

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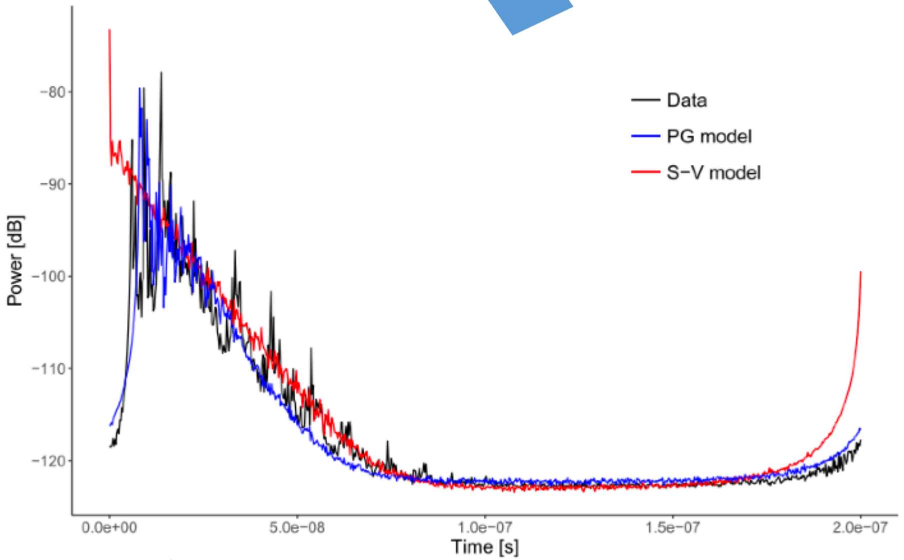
- Dimensionality of parameter space also a problem…



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Roadmap going ahead…

Background + challenges for SBI

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Snapshot 1:
Multi-fidelity methods for simulation-based inference (NeurIPS?, 2025)

Roadmap going ahead…

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Multi-fidelity methods for simulation-based inference (NeurIPS?, 2025)

Snapshot 2:
Cost-aware methods for simulation-based inference (AISTATS, 2025)

Roadmap going ahead…

Background + challenges for SBI

Snapshot 1:
Multi-fidelity methods for simulation-based inference (NeurIPS?, 2025)

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Cost-aware methods for simulation-based inference (AISTATS, 2025)

Snapshot 3:
Provably robust generalisation of Bayes for simulation-based inference (AISTATS Best Paper Award, 2022)

Any Questions?

Multilevel neural simulation-based inference

Paper: Hikida, Y., Bharti, A., Jeffrey, N. & **Briol, F-X** (2025). Multilevel neural simulation-based inference. arXiv:2506.06087. (to appear at NeurIPS?)

Code: https://github.com/yugahikida/multilevel-sbi

Challenge for SBI

Simulators can be really computationally expensive!

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- Most simulators used in SBI papers take only a few seconds (or less) to run.
- Even if a simulator takes only a few minutes, we typically need thousands of simulations!
- Simulators that take more time are currently out of reach of existing methods.

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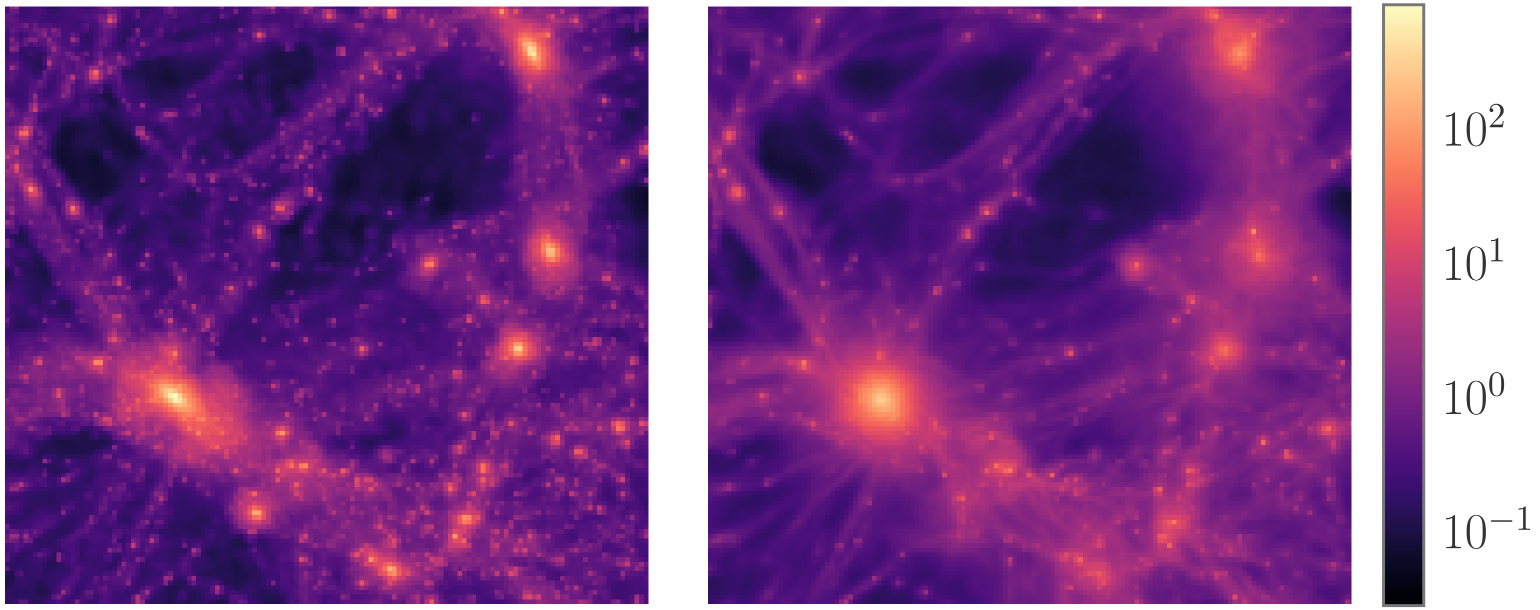
→ This leads to a form of model misspecification by design!

SBI for cosmology



Low-fidelity High-fidelity

Jeffrey, N., et al. (2025). Dark energy survey year 3 results: likelihood-free, simulation-based wCDM inference with neural compression of weak-lensing map statistics. *Monthly Notices of the Royal Astronomical Society*, *536*(2), 1303–1322.

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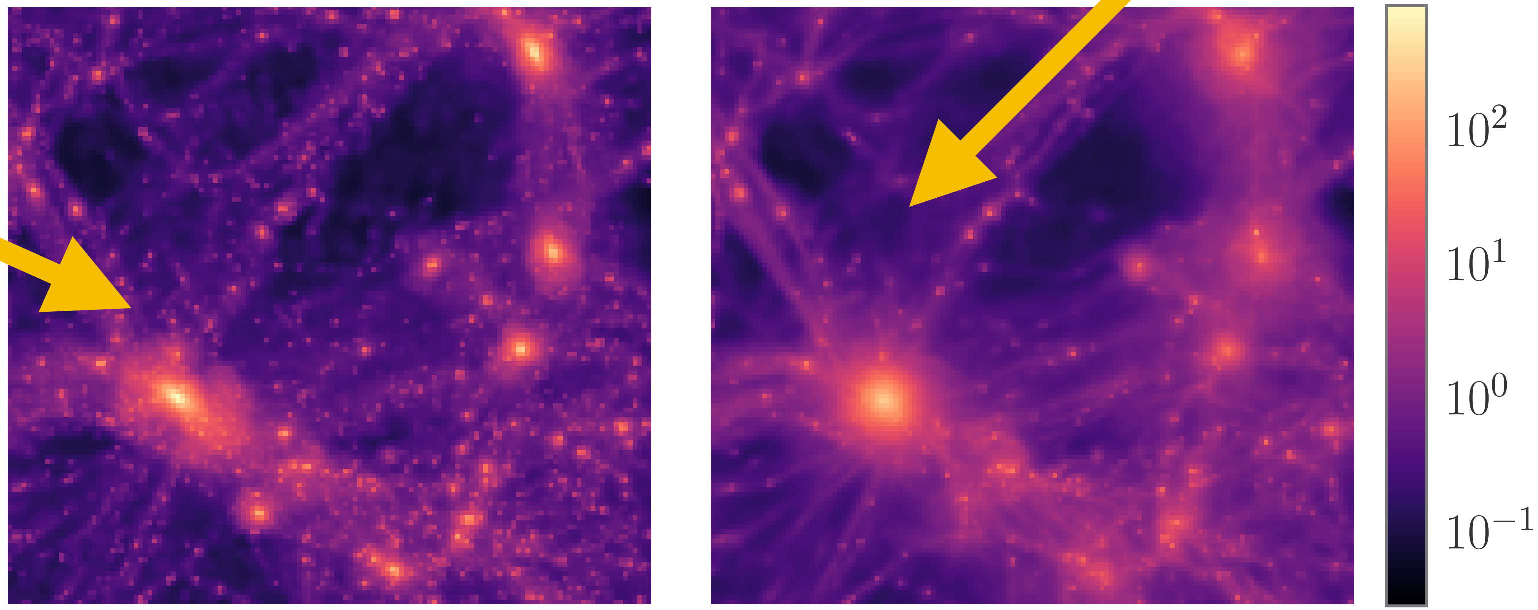
SBI for cosmology

Hydrodynamic simulations

Gravity-only N-body simulations



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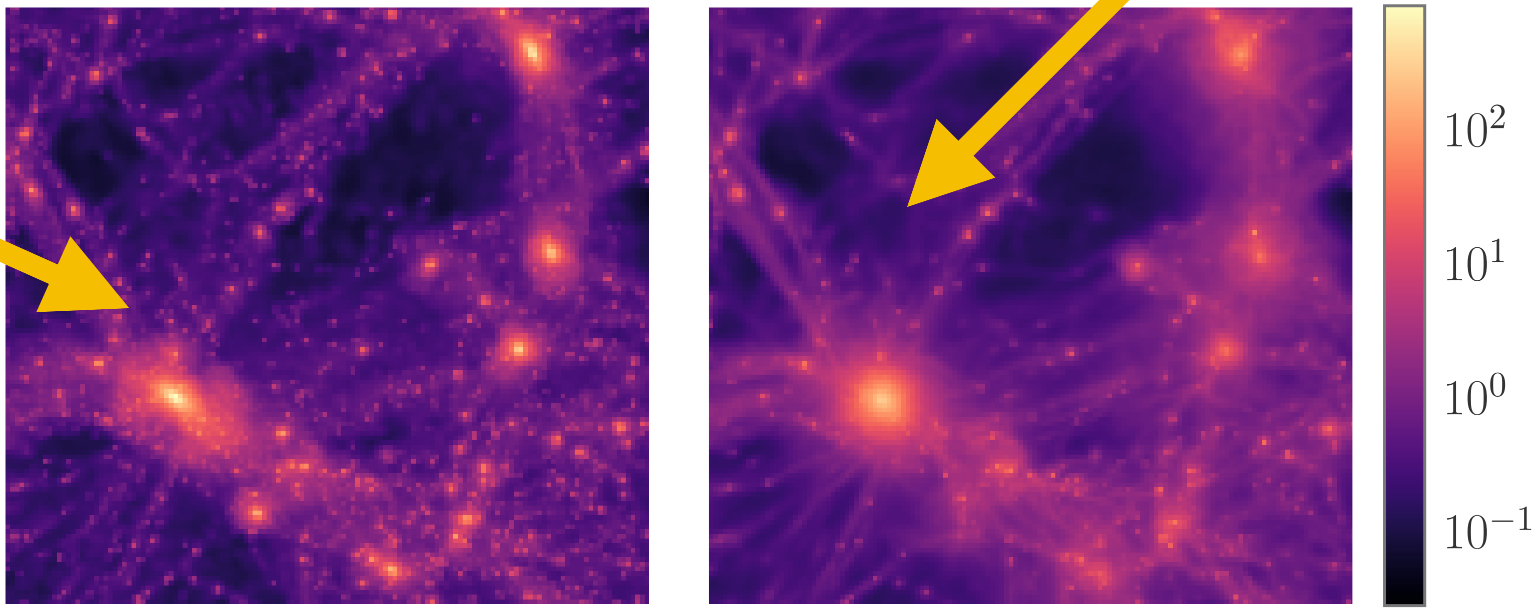

Low-fidelity High-fidelity

$\approx$ 100x more expensive!!

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Existing work on multi-fidelity in SBI

Many great works, but which are not specialised for neural-SBI:

Jasra, A., Jo, S., Nott, D., Shoemaker, C., & Tempone, R. (2019). Multilevel Monte Carlo in approximate Bayesian computation. *Stochastic Analysis and Applications*, *37*(3), 346–360.

Prescott, T. P., & Baker, R. E. (2020). Multifidelity approximate Bayesian computation. *SIAM-ASA Journal on Uncertainty Quantification*, *8*(1), 114–138.

Warne, D. J., Prescott, T. P., Baker, R. E., & Simpson, M. J. (2022). Multifidelity multilevel Monte Carlo to accelerate approximate Bayesian parameter inference for partially observed stochastic processes. *Journal of Computational Physics*, *469*, 111543.

Existing work on multi-fidelity in SBI

One very recent attempt, but no theory and critical issue with hyper parameter selection:

Krouglova, A. N., Johnson, H. R., Confavreux, B., Deistler, M., & Gonçalves, P. J. (2025). Multifidelity simulation-based inference for computationally expensive simulators. *arXiv:2502.08416*.

Existing work on multi-fidelity in SBI

Open problem: Rigorous and theoretically-grounded multi-fidelity for neural SBI!

Neural likelihood estimation (NLE)

- **Step 1:** train $q_\phi(\cdot \mid \theta)$ to approximate the likelihood using samples from the prior ($\theta_1, \ldots, \theta_n \sim p(\theta)$) and simulator ($x_i \sim p(\cdot \mid \theta_i)$):

$$\hat{\phi}_n := \arg\min_{\phi \in \Phi} \ell_{\text{NLE}}(\phi), \quad \ell_{\text{NLE}}(\phi) = -\frac{1}{n}\sum_{i=1}^{n} \log q_\phi(x_i \mid \theta_i) \approx -\mathbb{E}_{\theta \sim p(\theta)}[\mathbb{E}_{x \sim p(\cdot \mid \theta)}[\log q_\phi(x \mid \theta)]]$$

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- **Step 2:** Do Bayes with approximate likelihood!

$$p_{\mathrm{NLE}}(\theta\,|\,y_1, \ldots, y_n) \propto \prod_{i=1}^{n} q_{\hat{\phi}_n}(y_i\,|\,\theta)p(\theta)$$

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Typically the most **computationally expensive** step!!

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A better step 1?

$$\ell_{\text{NLE}}(\phi) = -\frac{1}{n}\sum_{i=1}^{n} \log q_\phi(x_i \,|\, \theta_i) \approx -\,\mathbb{E}_{\theta \sim p(\theta)}[\mathbb{E}_{x \sim p(\cdot|\theta)}[\log q_\phi(x \,|\, \theta)]]$$

Can we do this better/cheaper?!

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Can we do this better/cheaper?!

→ Yes, Multilevel Monte Carlo!

Giles, M. B. (2015). Multilevel Monte Carlo methods. *Acta Numerica*, *24*, 259–328.

Jasra, A., Law, K., & Suciu, C. (2020). Advanced Multilevel Monte Carlo Methods. *International Statistical Review*, *88*(3), 548–579.

Multilevel Monte Carlo

Suppose we have a $f_0, f_1, \ldots, f_L = f$ of increasing cost but also increasing accuracy. Then:

$$\mathbb{E}_{z \sim \mu}[f(z)]$$

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Multilevel Monte Carlo

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&\approx \frac{1}{n_0}\sum_{i=1}^{n_0} f_0(z_i^0) + \sum_{l=1}^{L}\left(\frac{1}{n_l}\sum_{i=1}^{n_l}\left(f_l\left(z_i^l\right) - f_{l-1}\left(z_i^l\right)\right)\right)
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Very cheap - can take n_0 large.

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\end{aligned}$$

Very cheap - can take n_0 large.

Very expensive - **cannot** take n_l large…. But low variance!

Multilevel NLE

$$-\mathbb{E}_{\theta\sim\pi}\left[\mathbb{E}_{x\sim\mathbb{P}_\theta}\left[\log q_\phi(x\,|\,\theta)\right]\right]$$

Multilevel NLE

Change of measure

$$-\mathbb{E}_{\theta\sim\pi}\left[\mathbb{E}_{x\sim\mathbb{P}_\theta}\left[\log q_\phi(x\,|\,\theta)\right]\right] = \mathbb{E}_{\theta\sim\pi,u\sim\mathbb{U}}\left[-\log q_\phi\left(G_\theta(u)\,|\,\theta\right)\right]$$

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→ This is now a joint expectation in the prior and $\mathbb{U}$!

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We can directly apply MLMC to it, where intermediate integrands are of the form:

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Multilevel neural SBI

Our ‘data’ is therefore:

$$\left\{ \theta_i^l, u_i^l, G_{\theta_i^l}^l\left(u_i^l\right), G_{\theta_i^l}^{l-1}\left(u_i^l\right) \right\} \quad \text{where} \quad \theta_i^l \sim \pi, u_i^l \sim \mathbb{U},$$

Multilevel neural SBI

Seed-matched!

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Our objective for step 1 is:

$$\ell_{\mathrm{ML-NLE}}(\phi) := \frac{1}{n_0}\sum_{i=1}^{n_0} f_\phi^0(u_i^0, \theta_i^0) + \sum_{l=1}^{L}\frac{1}{n_l}\sum_{i=1}^{n_l}\left(f_\phi^l(u_i^l, \theta_i^l) - f_\phi^{l-1}(u_i^l, \theta_i^l)\right)$$

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Note that we presented this for NLE, but the same could work for NPE, other scoring rules, etc…!

Challenges with training

$$\ell_{\mathrm{ML-NLE}}(\phi) := \frac{1}{n_0}\sum_{i=1}^{n_0} f_\phi^0(u_i^0, \theta_i^0) + \frac{1}{n_1}\sum_{i=1}^{n_1}\left(f_\phi^1(u_i^1, \theta_i^1) - f_\phi^0(u_i^1, \theta_i^1)\right)$$

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Contradictory gradients! This is a problem when we are close to stationarity and n_0/n_1 are small… The variance of the negative term is always large!!

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Contradictory gradients! This is a problem when we are close to stationarity and n_0/n_1 are small… The variance of the negative term is always large!!

We fix the issue by normalising gradients so that these two terms have the same magnitude, and by projecting onto each other's normal planes, which stabilises training.

Bound on the variance

Under some mild assumptions, we get:

$$\text{Var}\left[\ell_{\text{ML-NLE}}(\phi)\right] \leq \frac{K_0(\phi)}{n_0}\left(\|G^0\|^4_{W^{1,4}(\pi\times\mathbb{U})} + 1\right) + \sum_{l=1}^{L}\frac{K_l(\phi)}{n_l}\left(\|G^l - G^{l-1}\|^2_{W^{1,4}(\pi\times\mathbb{U})} + 1\right)$$

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Large!

Small!

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Complexity of low-fidelity generator - large!

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Complexity of other integrands - small!

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3) The surrogate $q_\phi(\cdot \mid \theta)$ has a Lipschitz gradient locally, and does not blow up too fast.

Bound on the variance

Can use this to determine optimal samples per level!

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Simulations per level

Under some mild regularity conditions, we can find the optimal number of simulations per level assuming we have a maximum computational budget of C_{budget}:

$$n_0^\star \propto \frac{C_{\text{budget}}}{\sqrt{C_0}} \sqrt{\|G^0\|^4_{W^{1,4}(\pi \times \mathbb{U})} + 1}, \qquad n_l^\star \propto \frac{C_{\text{budget}}}{\sqrt{C_l + C_{l+1}}} \sqrt{\|G^l - G^{l-1}\|^2_{W^{1,4}(\pi \times \mathbb{U})} + 1}\,.$$

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The more 'complex' the generator (or the difference in generators), the more simulations we need.

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The larger the cost of simulations at this level, the less simulations we can afford.

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Note that these expressions contain a lot of quantities we may not know a-priori, but it is still indicative and helpful for selecting which simulations to run in practice.

G-and-k distribution

$$G_\theta^l(u) = \theta_1 + \theta_2 \left(1 + 0.8 \left(\frac{1 - \exp(-\theta_3 z_l(u))}{1 + \exp(-\theta_3 z_l(u))} \right) \right) \left(1 + z_l(u)^2\right)^{\log(\theta_4)} z_l(u),$$

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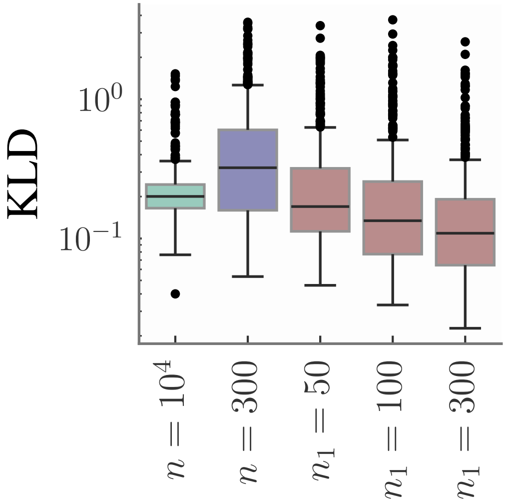

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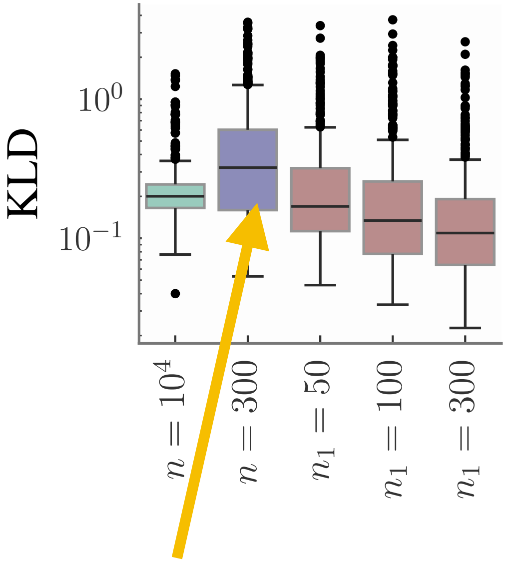

High-fidelity only:
too few simulations!

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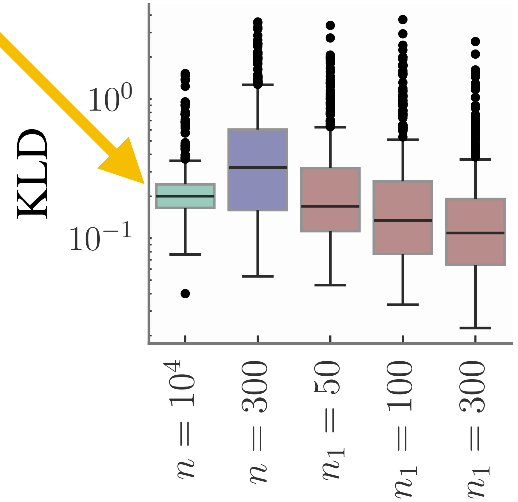
Low-fidelity only:
Many simulations,
but low quality!



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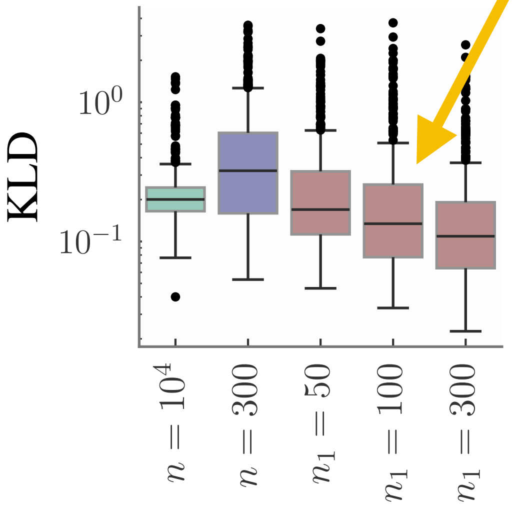
ML-NLE: both many simulations and high quality!



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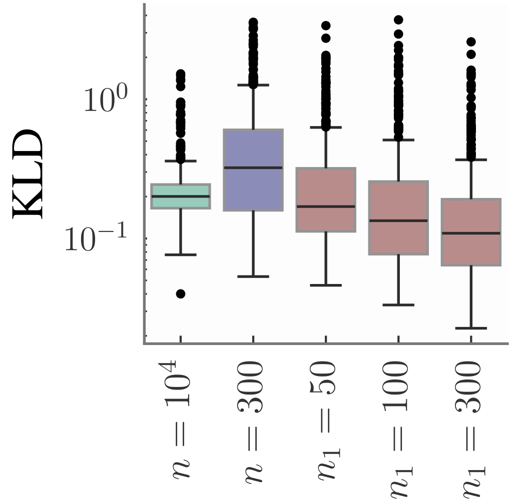

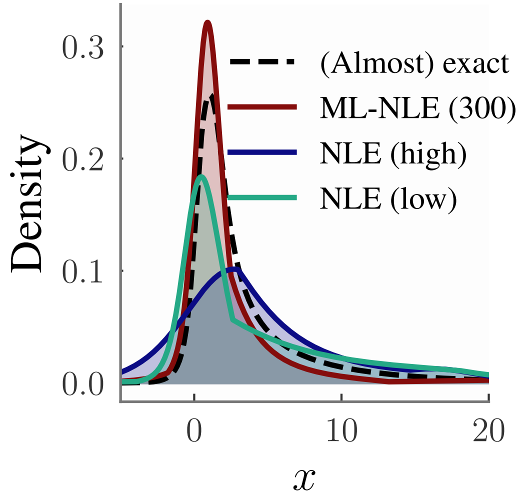

Prangle, D. (2020). gk: An R Package for the g-and-k and generalised g-and-h distributions. The R Journal, 12(1):7.

G-and-k distribution

$$G^l_\theta(u) = \theta_1 + \theta_2 \left(1 + 0.8\left(\frac{1-\exp(-\theta_3 z_l(u))}{1+\exp(-\theta_3 z_l(u))}\right)\right)\left(1 + z_l(u)^2\right)^{\log(\theta_4)} z_l(u),$$

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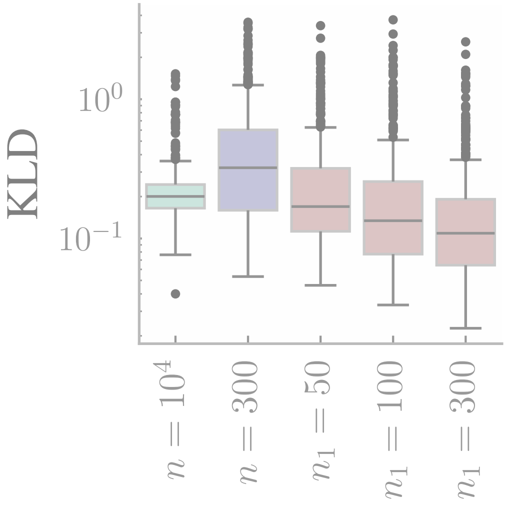

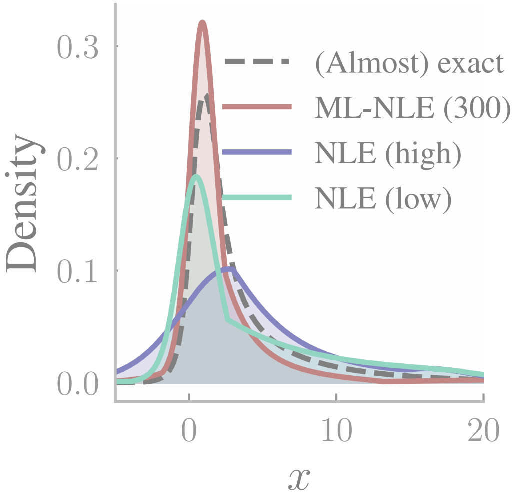

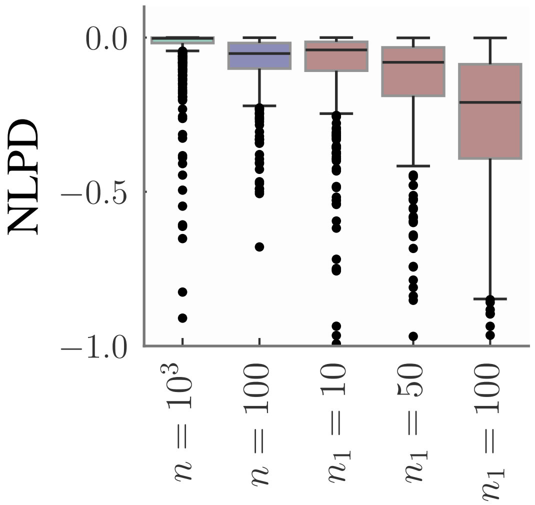

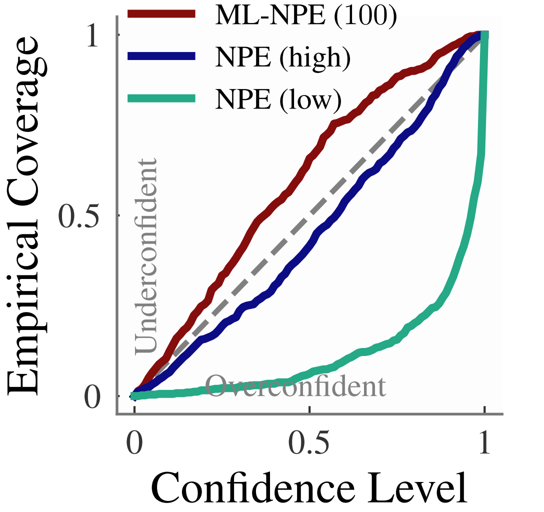

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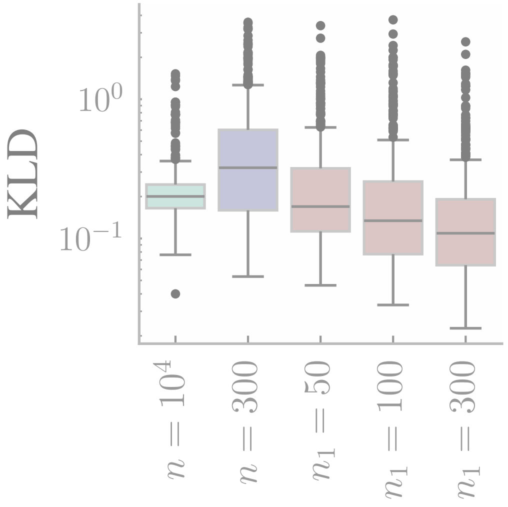

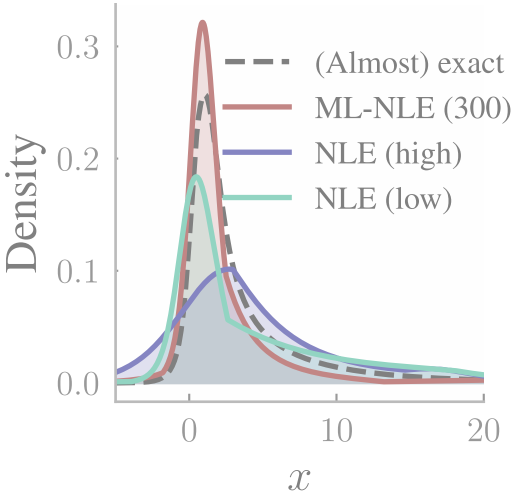

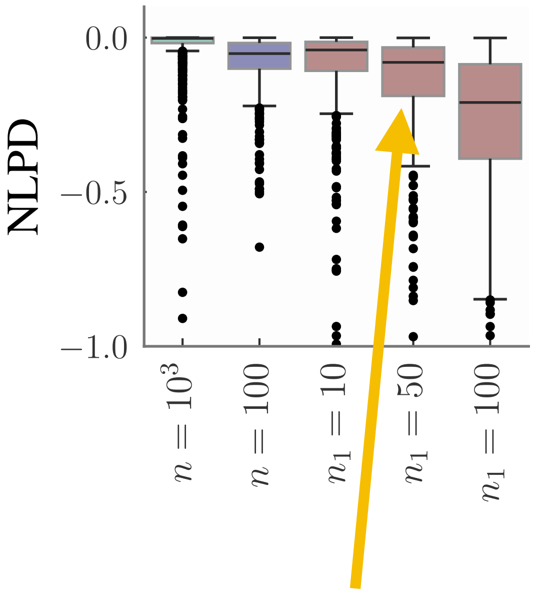

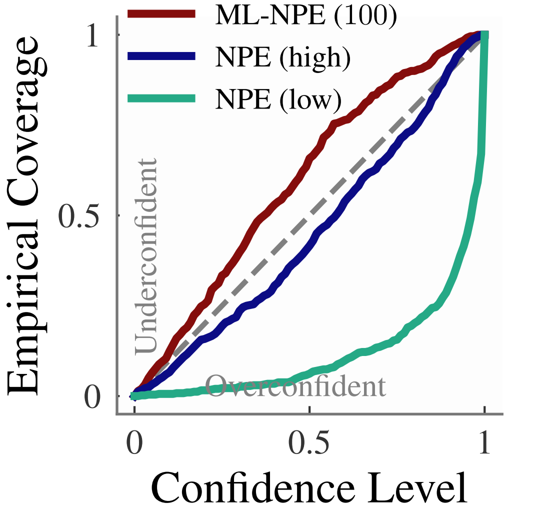

ML-NPE: Similar conclusion!

Prangle, D. (2020). gk: An R Package for the g-and-k and generalised g-and-h distributions. The R Journal, 12(1):7.

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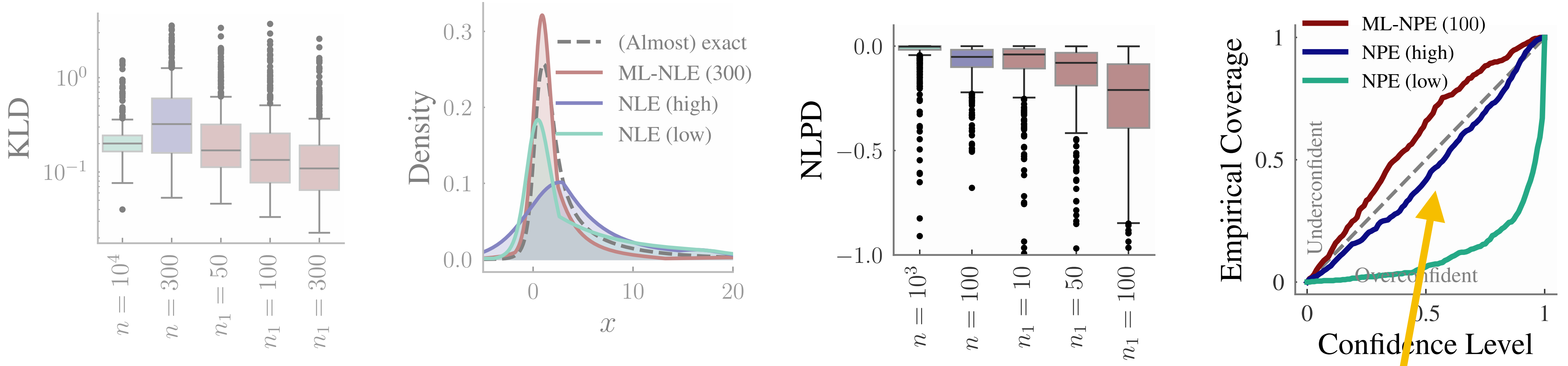
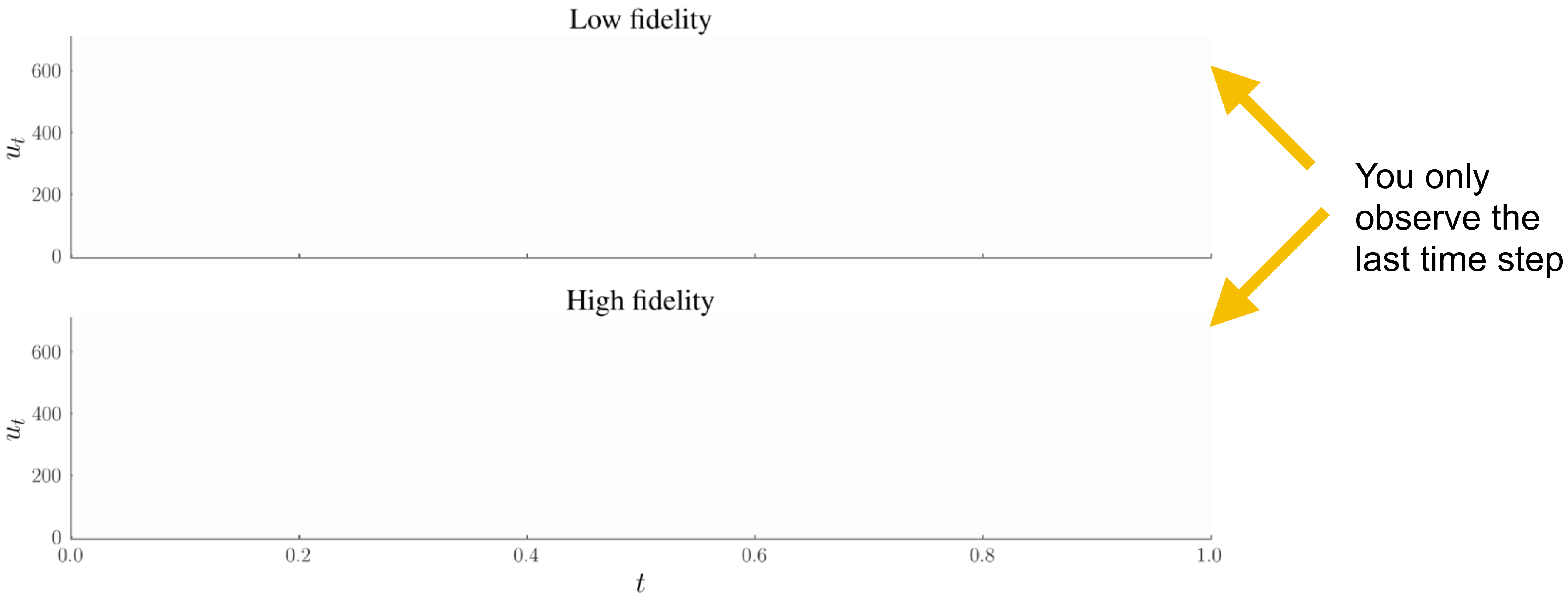


Coverage slightly cautious

Prangle, D. (2020). gk: An R Package for the g-and-k and generalised g-and-h distributions. The R Journal, 12(1):7.

Toggle-switch models for genes (d=1, p=7)



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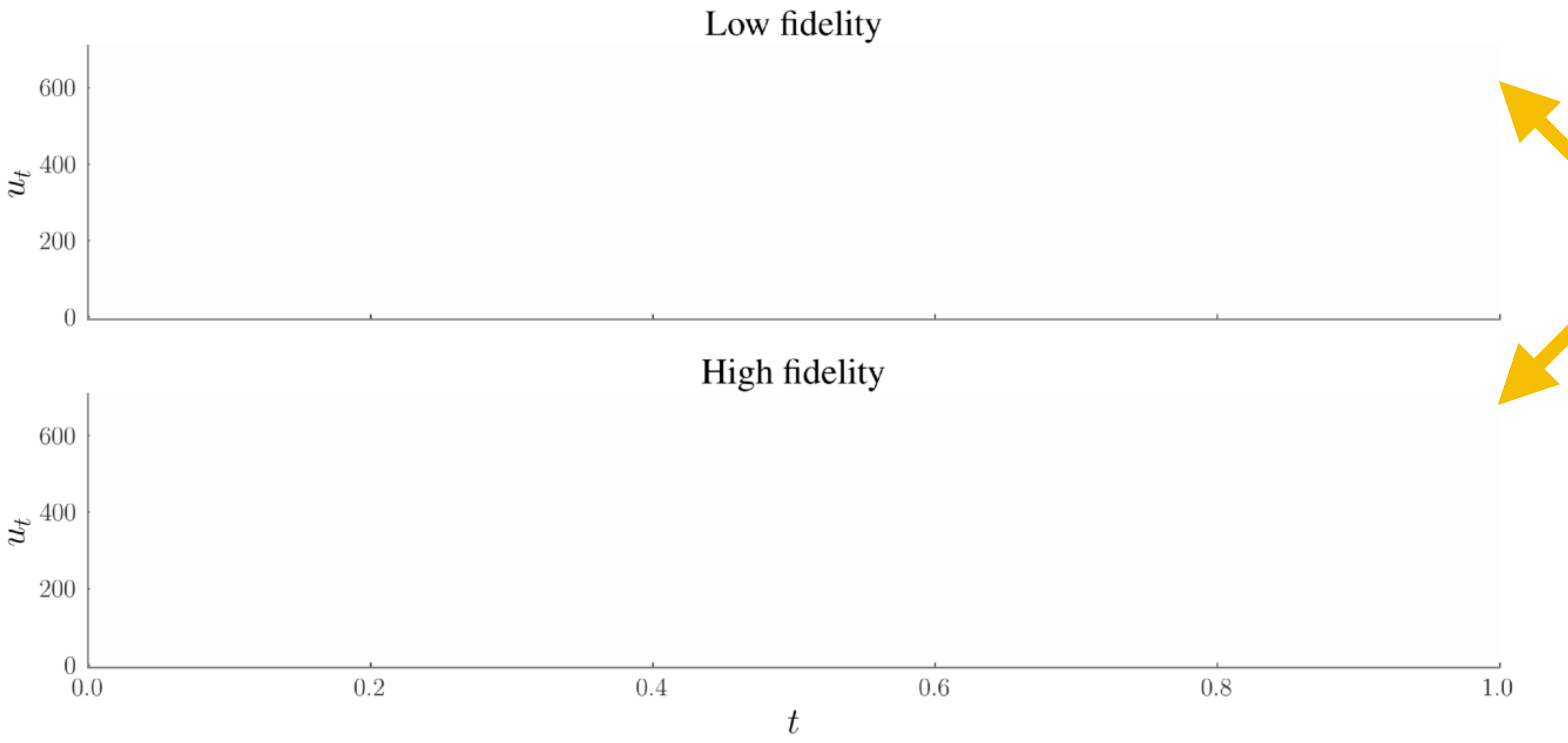

You only observe the last time step

Bonassi, F. V., You, L., & West, M. (2011). Bayesian learning from marginal data in bionetwork models. *Statistical Applications in Genetics and Molecular Biology*, *10*(1).

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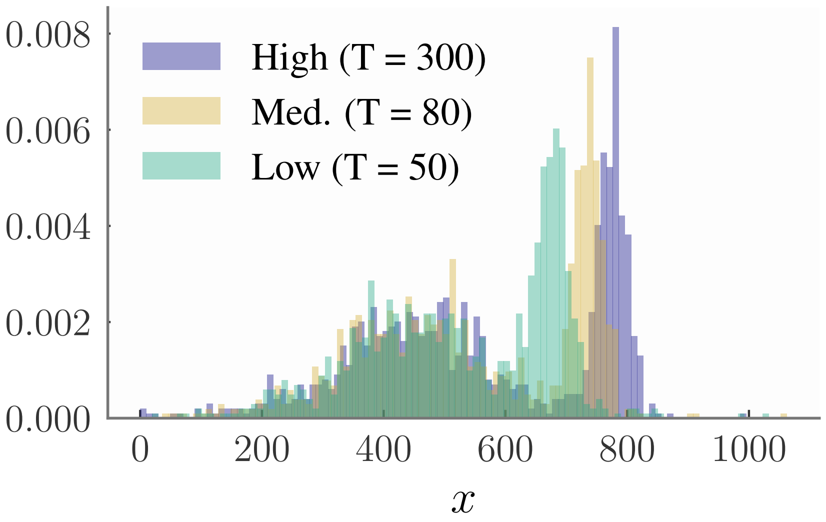

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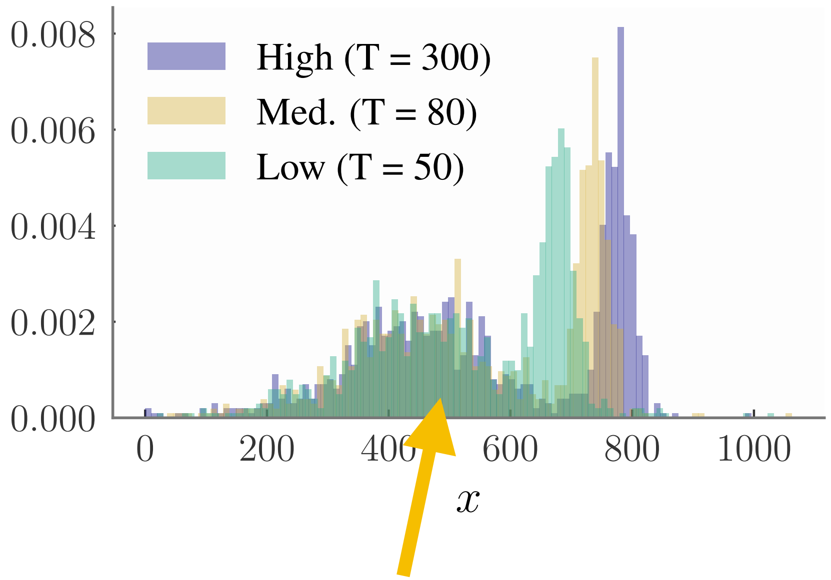

Observations bi-modal, with second mode only well approximated for high-fidelity levels

Bonassi, F. V., You, L., & West, M. (2011). Bayesian learning from marginal data in bionetwork models. *Statistical Applications in Genetics and Molecular Biology*, *10*(1).

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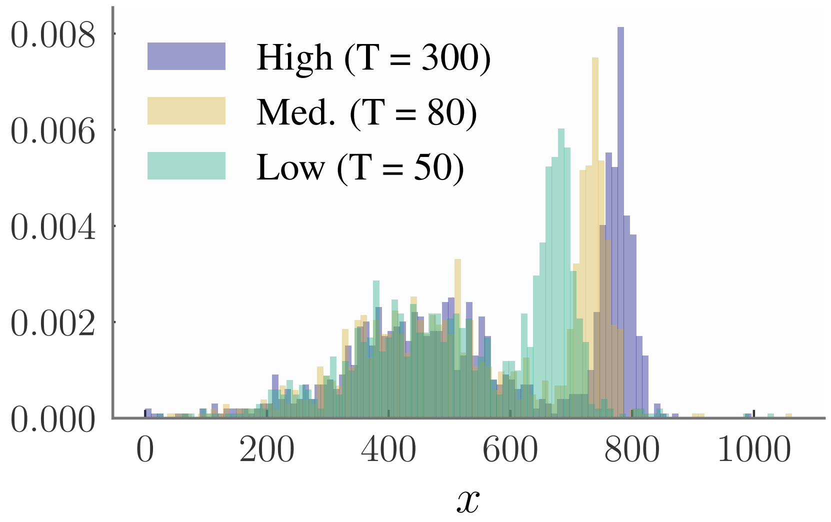

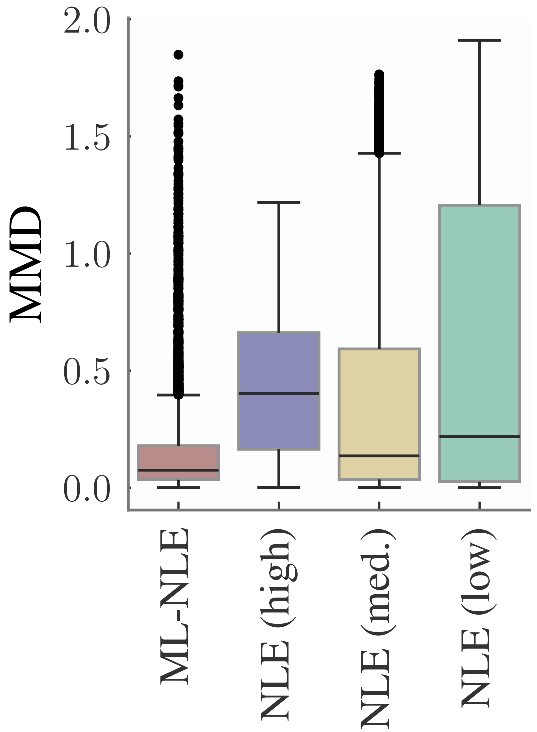

$n_0 = 10000$
$n_1 = 500$
$n_2 = 300$

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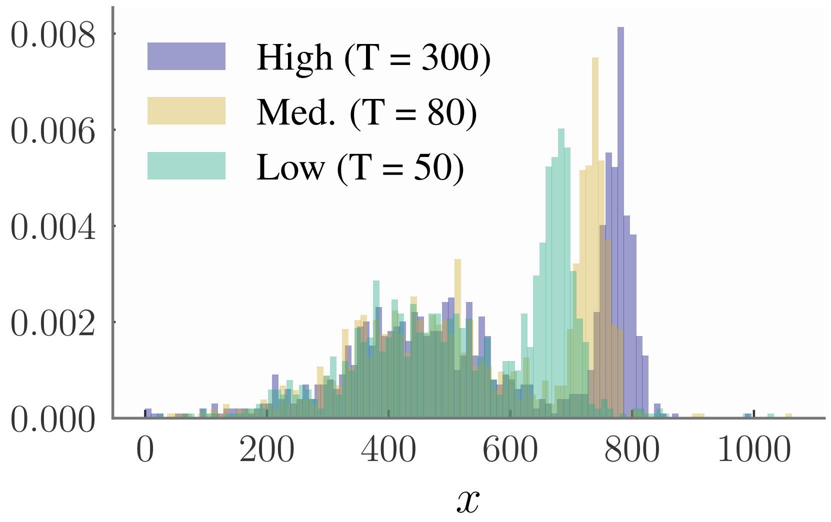

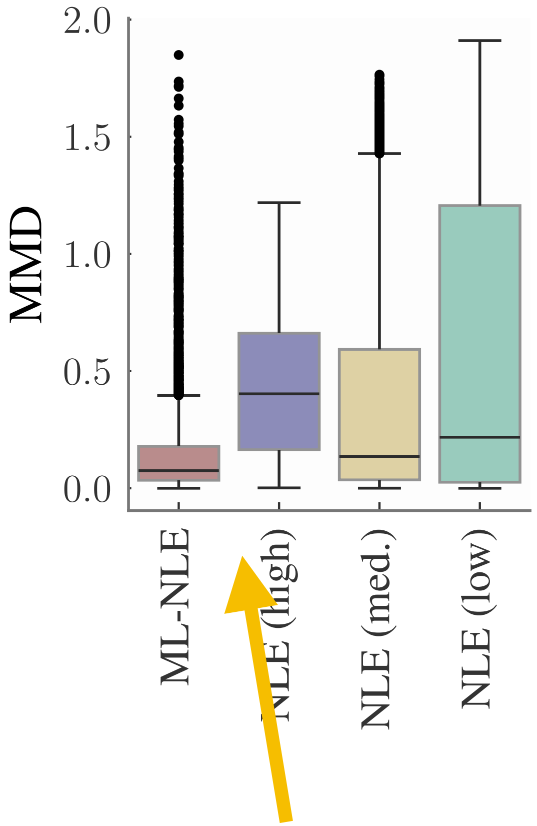

$n_0 = 10000$
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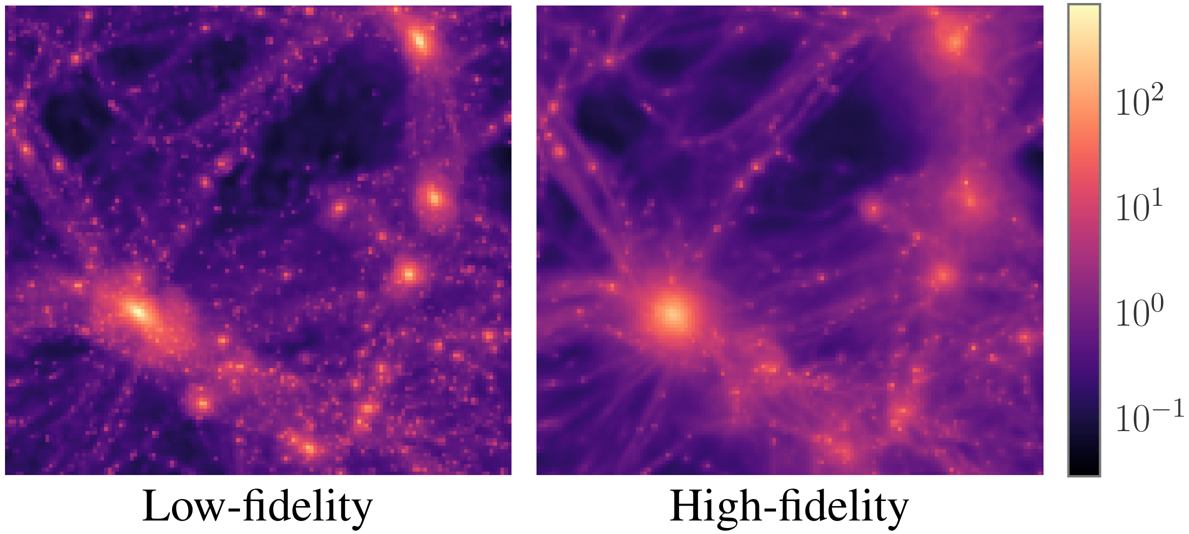
ML-NLE benefits from low-fidelity simulations for first mode but also from high-fidelity simulations for second mode

Bonassi, F. V., You, L., & West, M. (2011). Bayesian learning from marginal data in bionetwork models. *Statistical Applications in Genetics and Molecular Biology*, *10*(1).

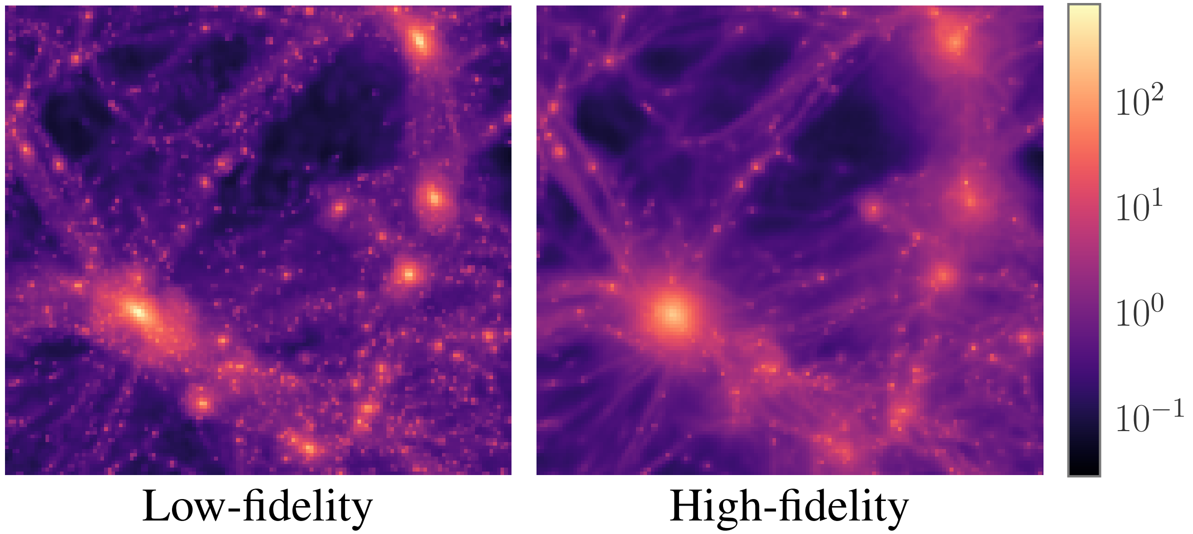
Back to cosmology…. (d=39, p=1)

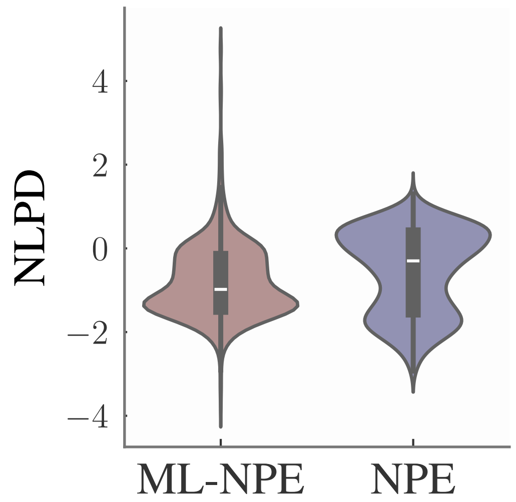


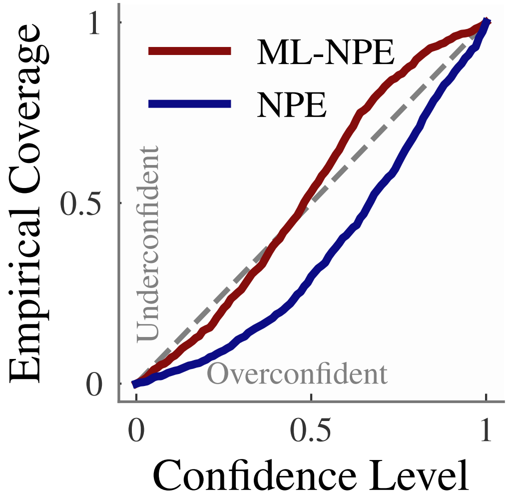
NPE: $n = 20$ (all high fidelity!)

ML-NPE: $n_0 = 20, \quad n_1 = 980$

Back to cosmology…. (d=39, p=1)







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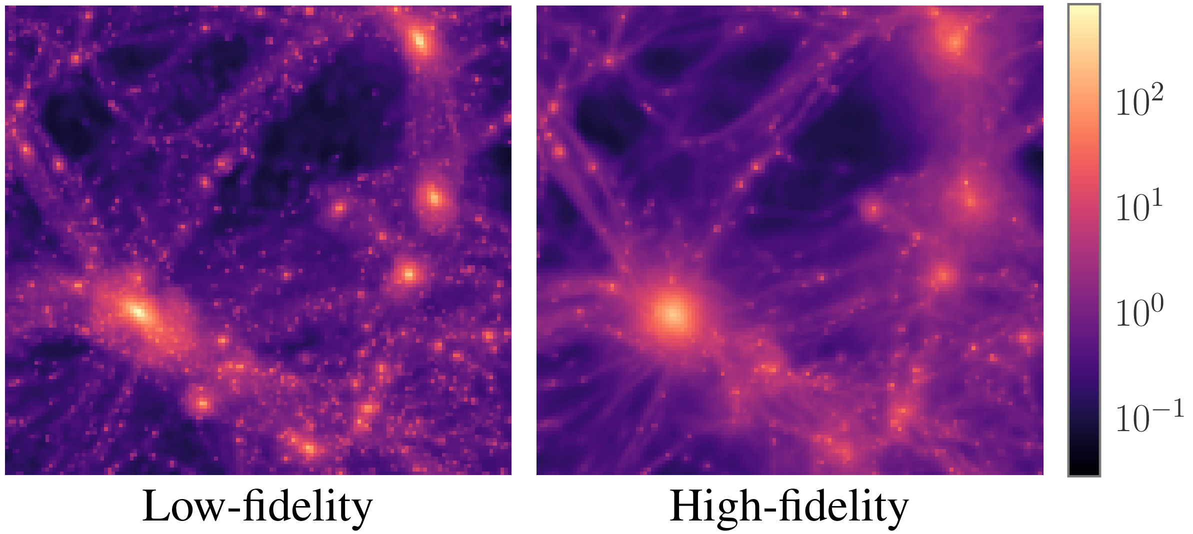

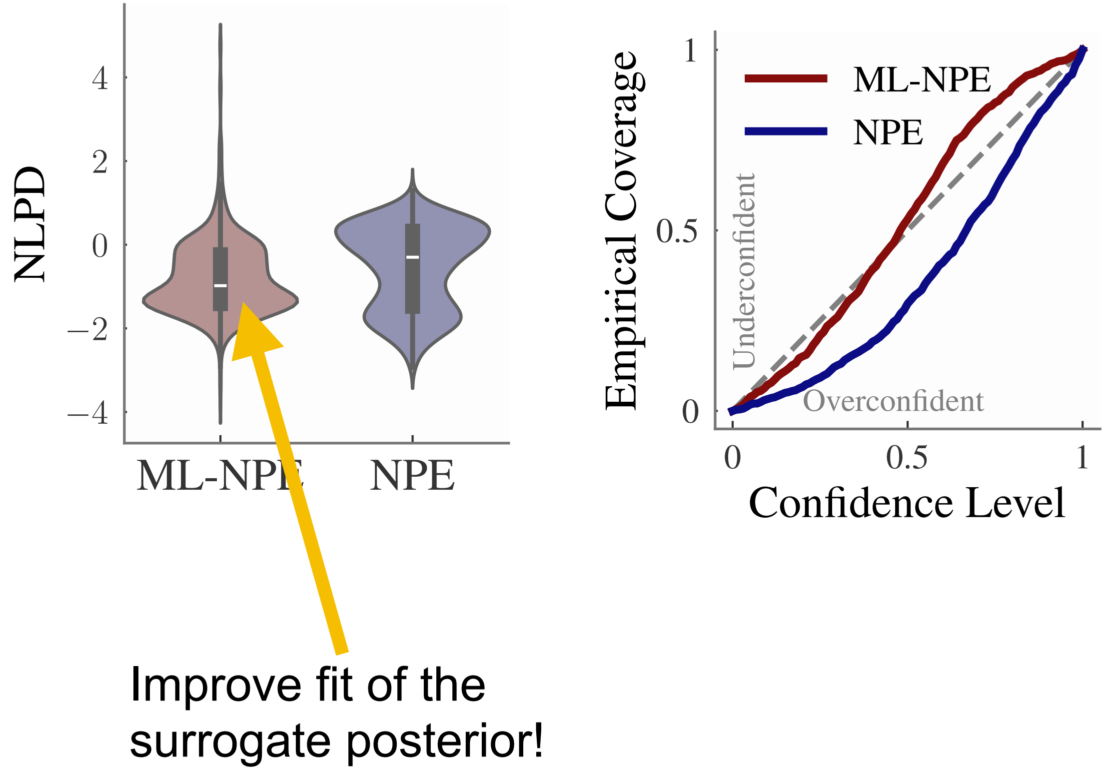

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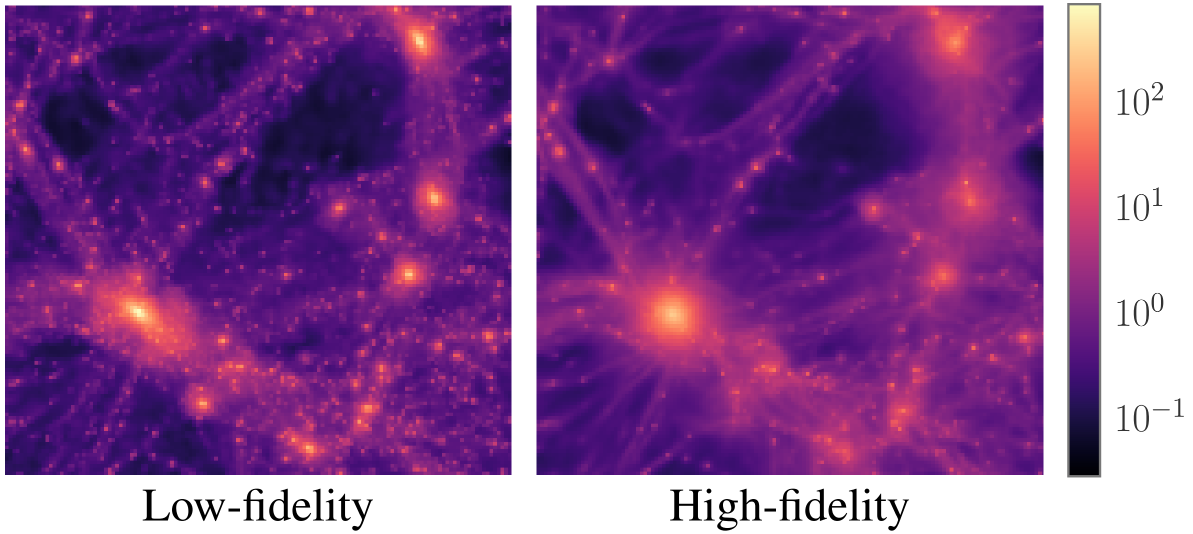

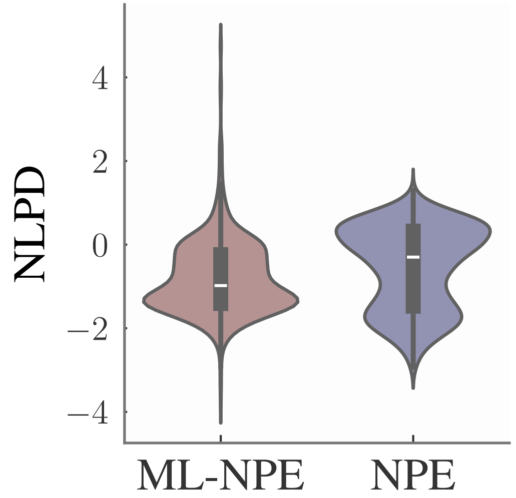

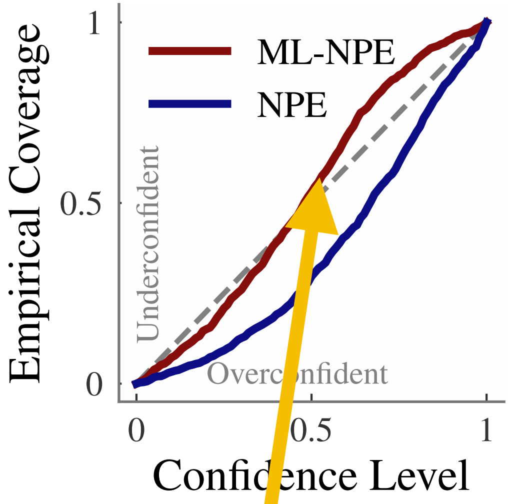

Improved calibration!

NPE: $n = 20$ (all high fidelity!)

ML-NPE: $n_0 = 20, \quad n_1 = 980$

Any Questions?

Paper: Hikida, Y., Bharti, A., Jeffrey, N. & Briol, F-X (2025). Multilevel neural simulation-based inference. arXiv:2506.06087 (to appear at NeurIPS?).

Code: https://github.com/yugahikida/multilevel-sbi

Cost-aware simulation-based inference



Paper: Bharti, A., Huang, D., Kaski, S., & **Briol, F.-X.** (2025). Cost-aware simulation-based inference. International Conference on Artificial Intelligence and Statistics, 28–36.

Code: https://github.com/huangdaolang/cost-aware-sbi

Challenge for SBI

Simulators can be really computationally expensive!

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However we may not have an easy way to obtain low-fidelity simulators….

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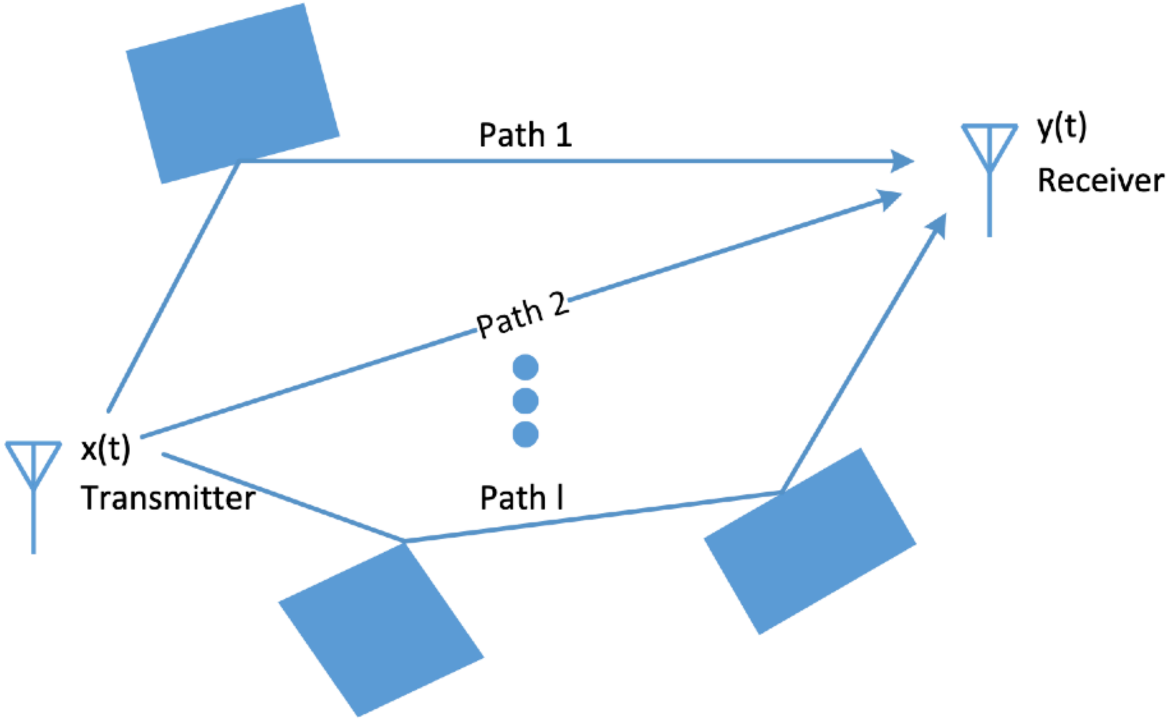
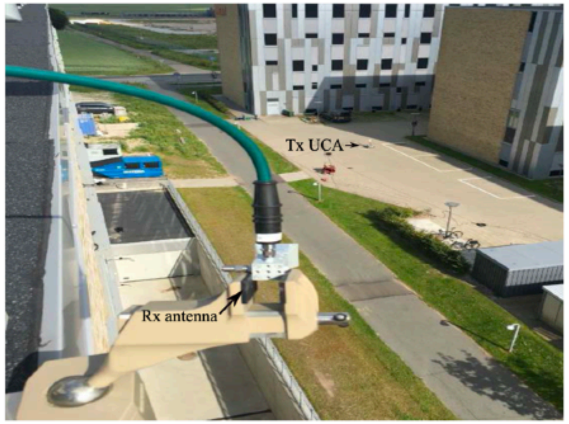
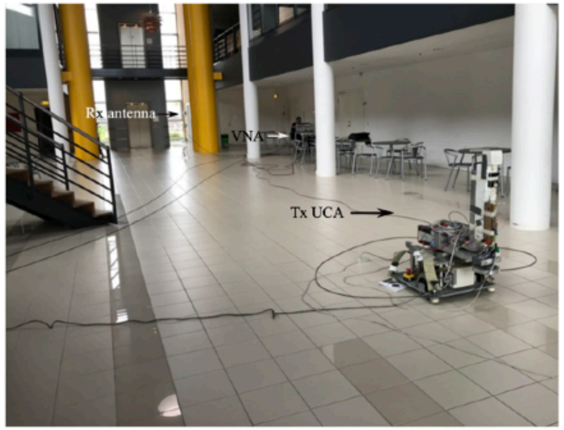
However we may not have an easy way to obtain low-fidelity simulators….

→ We can adjust our sampling to sample less often from expensive parameterisations!

SBI for radio-propagation







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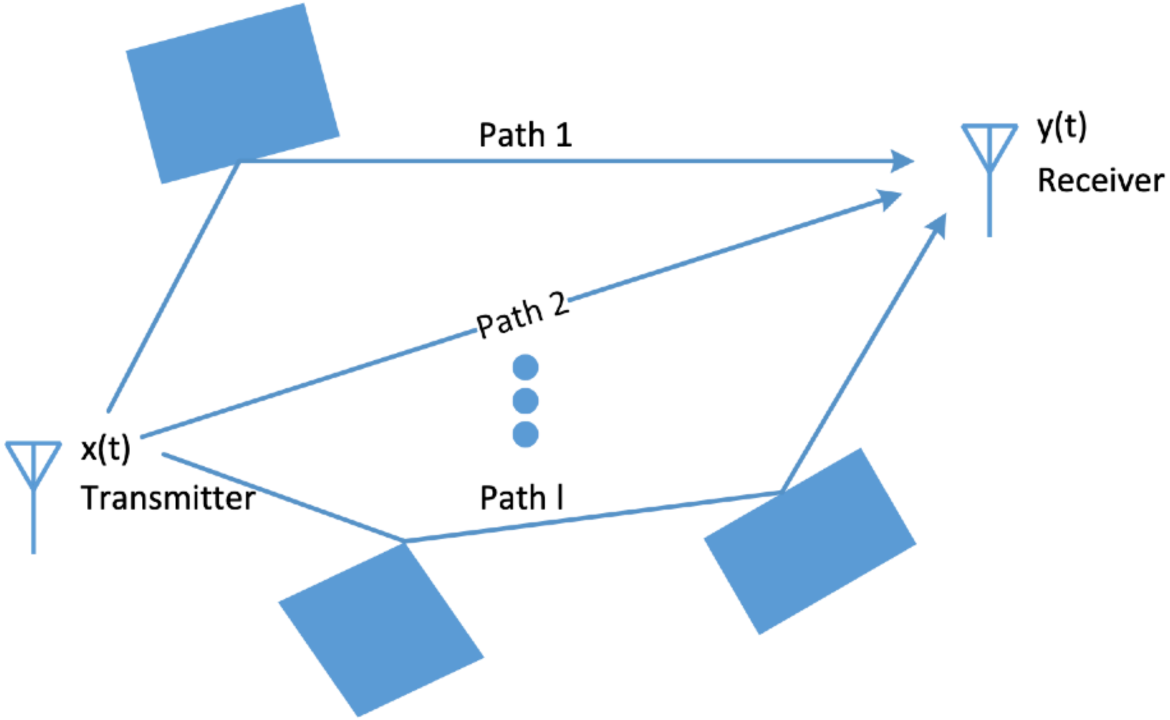

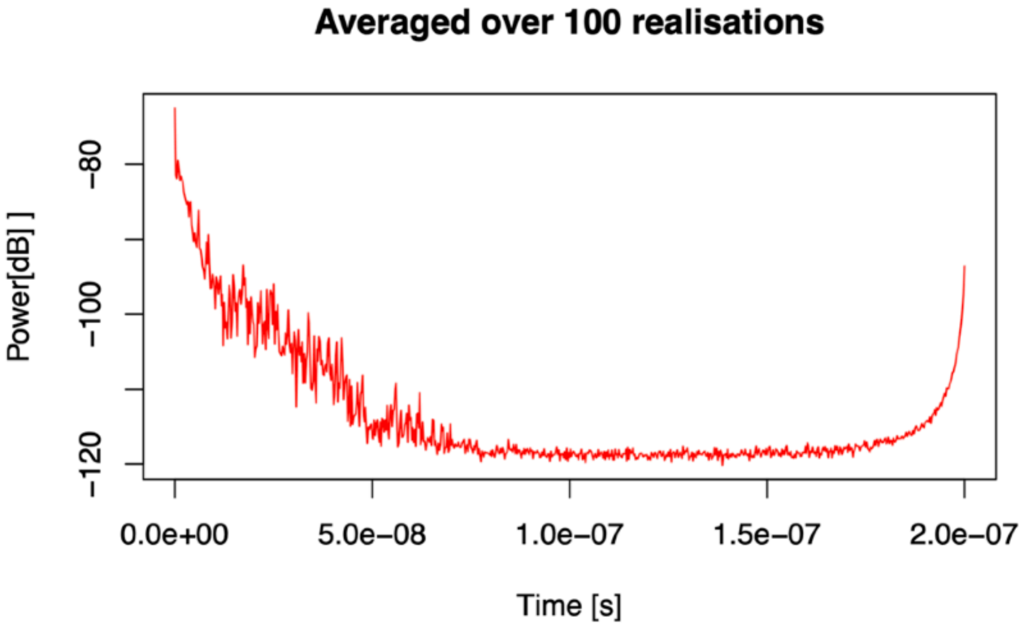

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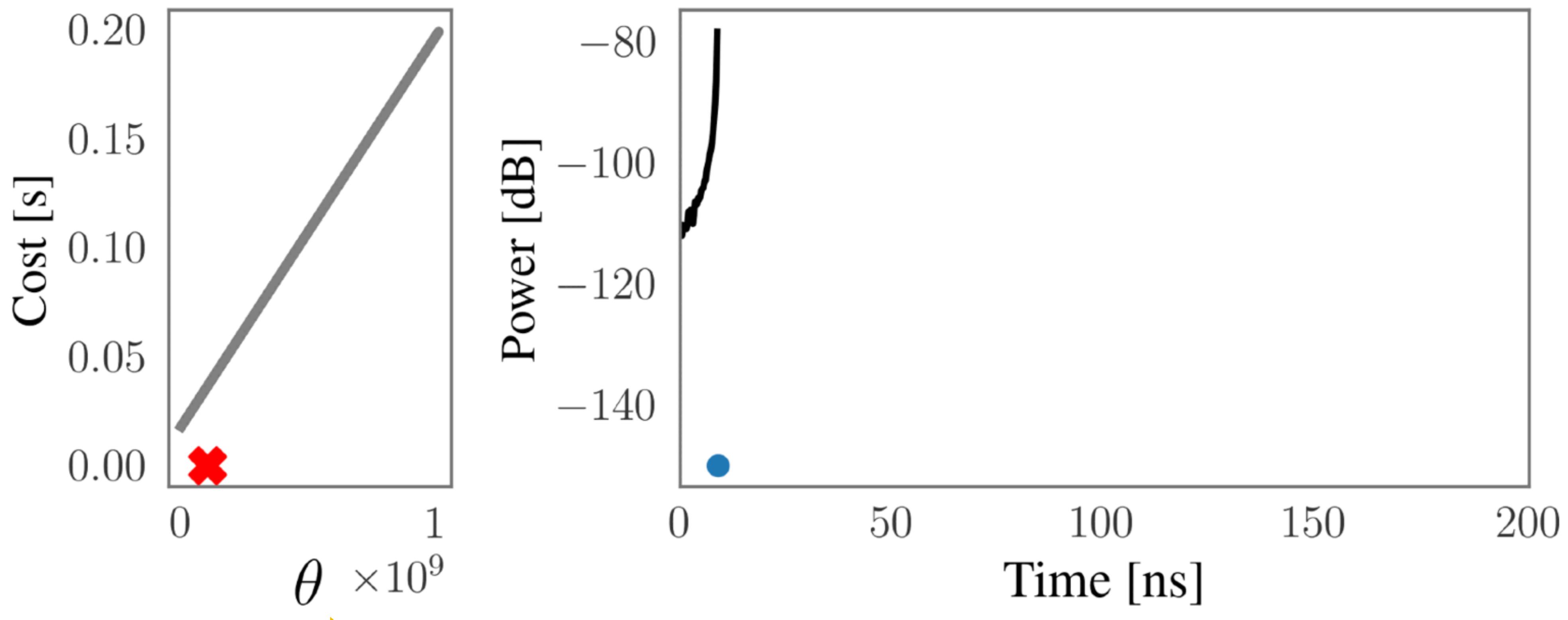
The cost of simulations is not constant…



Rate parameter of a Poisson process!

[https://fxbriol.github.io/images/ca-SBI.mp4]

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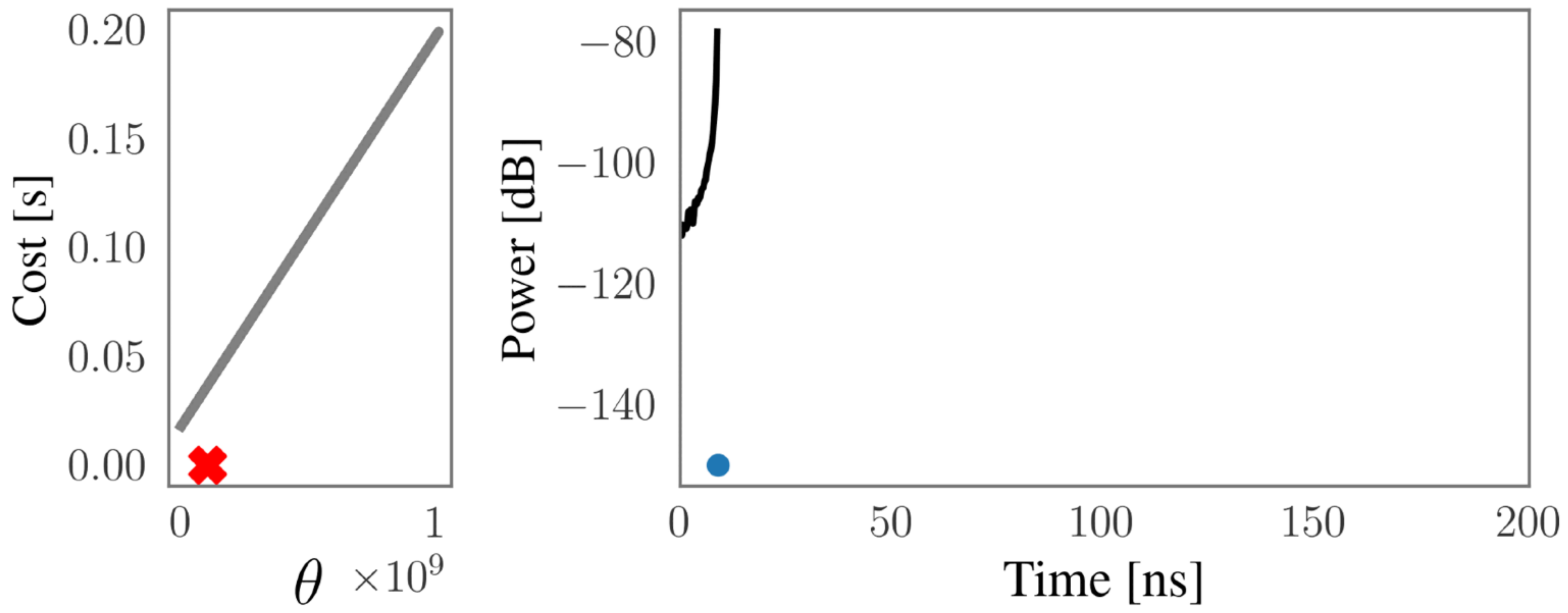

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Neural likelihood estimation (NLE)

- **Step 1:** train a conditional density model $q_\phi(\,\cdot\,|\,\theta)$ to approximate the likelihood using samples from the prior ($\theta_1, \ldots, \theta_n \sim p(\theta)$) and simulator ($x_i \sim p(\,\cdot\,|\,\theta_i)$):

$$\hat{\phi}_n := \arg\min_{\phi \in \Phi} \ell_{\text{NLE}}(\phi), \quad \ell_{\text{NLE}}(\phi) = -\frac{1}{n}\sum_{i=1}^{n} \log q_\phi(x_i\,|\,\theta_i) \approx -\mathbb{E}_{\theta\sim p(\theta)}[\mathbb{E}_{x\sim p(x|\theta)}[\log q_\phi(x\,|\,\theta)]]$$

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- **Step 2:** Do Bayes with approximate likelihood!

$$p_{\text{NLE}}(\theta \mid y_1, \ldots, y_n) \propto \prod_{i=1}^{n} q_{\hat{\phi}_n}(y_i \mid \theta) p(\theta)$$

A cheaper step 1?

$$\ell_{\mathrm{NLE}}(\phi) = -\frac{1}{n}\sum_{i=1}^{n} \log q_\phi(x_i \,|\, \theta_i) \approx -\mathbb{E}_{\theta\sim p(\theta)}[\mathbb{E}_{x\sim p(\cdot|\theta)}[\log q_\phi(x \,|\, \theta)]]$$

Can we do this better/cheaper?!

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Idea:

- Let's make use of the cost function $c : \Theta \to \mathbb{R}$.

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Can we do this better/cheaper?!

<u>Idea:</u>

- Let's make use of the cost function $c : \Theta \to \mathbb{R}$.
- We can try to sample less often in expensive regions ……
  but we still want to target the right objective.

Importance sampling

$$\mu = \int_{\Theta} f(\theta)\pi(\theta)d\theta$$

Importance sampling

$$\mu = \int_{\Theta} f(\theta)\pi(\theta)d\theta \quad = \int_{\Theta} f(\theta)\frac{\pi(\theta)}{\tilde{\pi}(\theta)}\tilde{\pi}(\theta)d\theta$$

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$$\approx \sum_{i=1}^{N} w(\theta_i)f(\theta_i) \qquad \theta_1, \ldots, \theta_N \sim \tilde{\pi}$$

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Question: How do you pick the importance distribution?

Cost-aware importance sampling

$$\tilde{\pi}_g(\theta) \propto \frac{\pi(\theta)}{g(c(\theta))},$$

Cost-aware importance sampling

$$\tilde{\pi}_g(\theta) \propto \frac{\boxed{\pi(\theta)}}{g(c(\theta))},$$

We want a distribution similar to our target π

Cost-aware importance sampling

$$\tilde{\pi}_g(\theta) \propto \frac{\pi(\theta)}{g(\boxed{c(\theta)})},$$

We do not want to sample often where the cost is large!

Cost-aware importance sampling

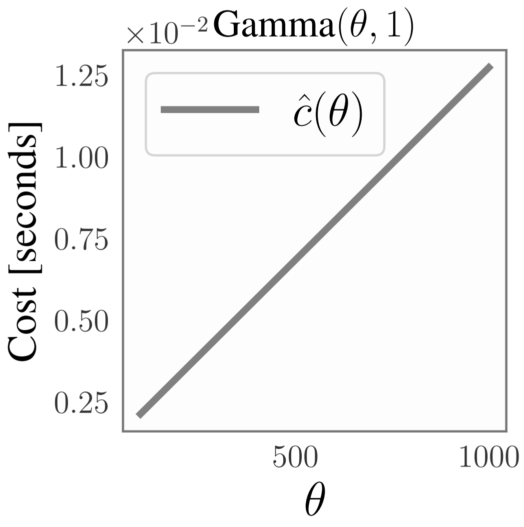
$$\tilde{\pi}_g(\theta) \propto \frac{\pi(\theta)}{\boxed{g}(c(\theta))},$$

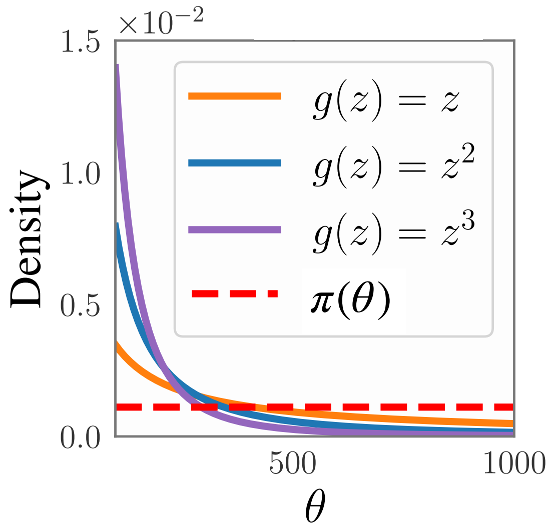
$g : (0,\infty) \to (0,\infty)$ taken to be non-decreasing.

Represents how much we dislike 'expensive' parameters!

Cost-aware importance sampling

$$\tilde{\pi}_g(\theta) \propto \frac{\pi(\theta)}{g(c(\theta))},$$





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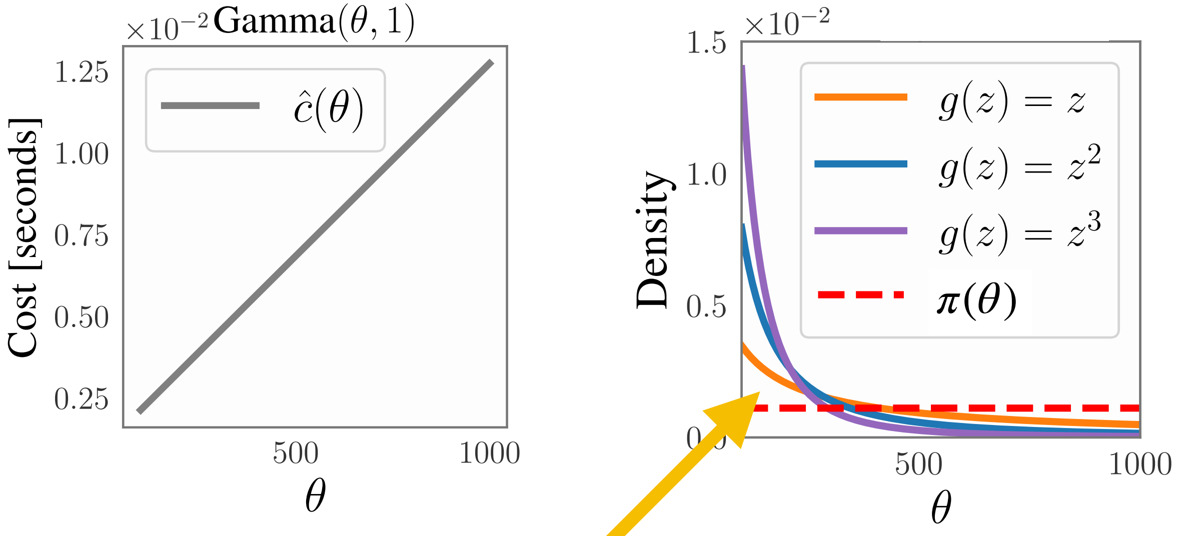

Upweight cheap region!

Cost-aware importance sampling

$$w(\theta) = \frac{1}{N}\frac{\pi(\theta)}{\tilde{\pi}_g(\theta)} = \frac{B\pi(\theta)g(c(\theta))}{N\pi(\theta)} \propto g(c(\theta))$$

Through $\tilde{\pi}_g$, we sample less often from expensive regions, so we need to up-weight expensive samples.

Cost-aware importance sampling

$$w(\theta) = \frac{1}{N}\frac{\pi(\theta)}{\tilde{\pi}_g(\theta)} = \frac{B\pi(\theta)g(c(\theta))}{N\pi(\theta)} \propto g(c(\theta))$$

$$w_{\mathrm{Ca}}(\theta_i) = \frac{w(\theta_i)}{\sum_{j=1}^{n} w(\theta_j)} = \frac{g(c(\theta_i))}{\sum_{j=1}^{n} g(c(\theta_j))}$$

We use SNIS weights

$$\mu = \int_{\Theta} f(\theta)\pi(\theta)d\theta \approx \sum_{i=1}^{n} w_{\mathrm{Ca}}(\theta_i)f(\theta_i) = \hat{\mu}_n^{\mathrm{Ca}}$$

Sampling from the cost-aware proposal

- We can use rejection sampling!

Sampling from the cost-aware proposal

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Repeat until n samples are accepted:

1. Sample $\theta^\star \sim \pi(\theta)$.
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Proposition: Assume $g_{\min} := \inf_{\theta \in \Theta} g(c(\theta)) > 0$. Then

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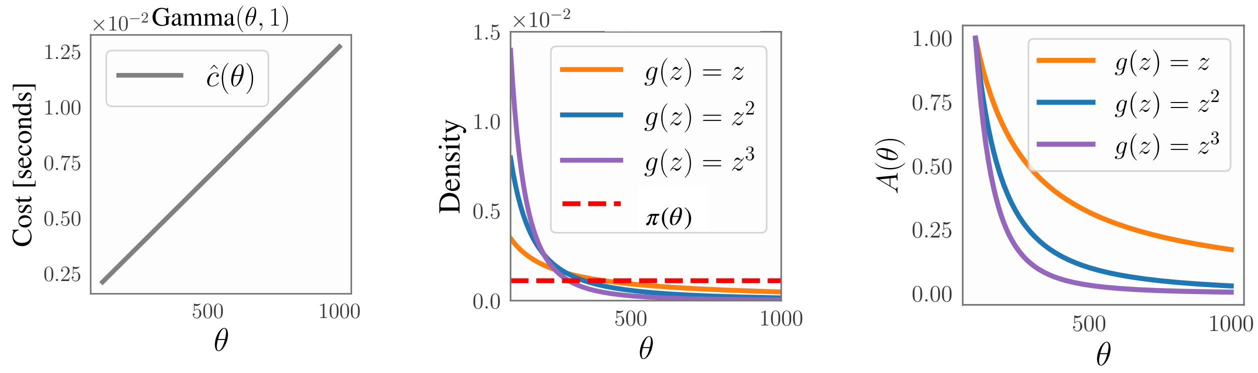

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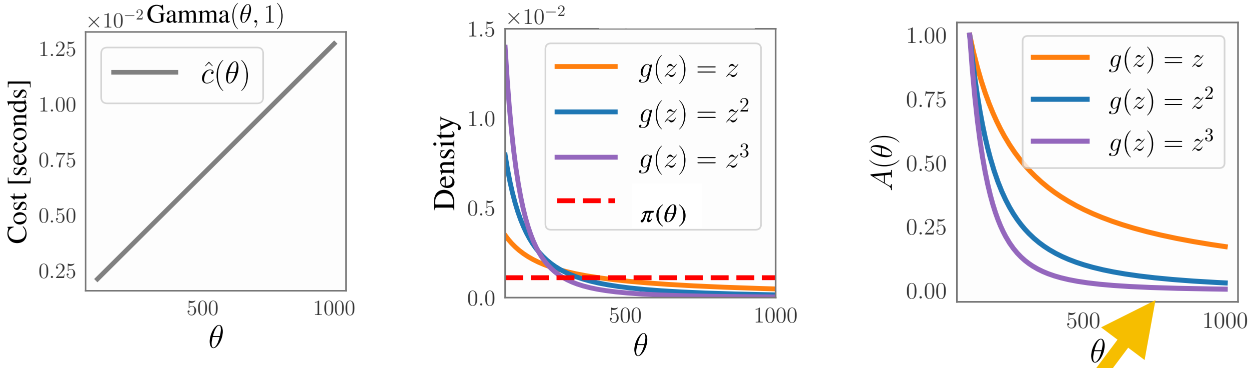

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Being cost-averse
decreases acceptance prob!

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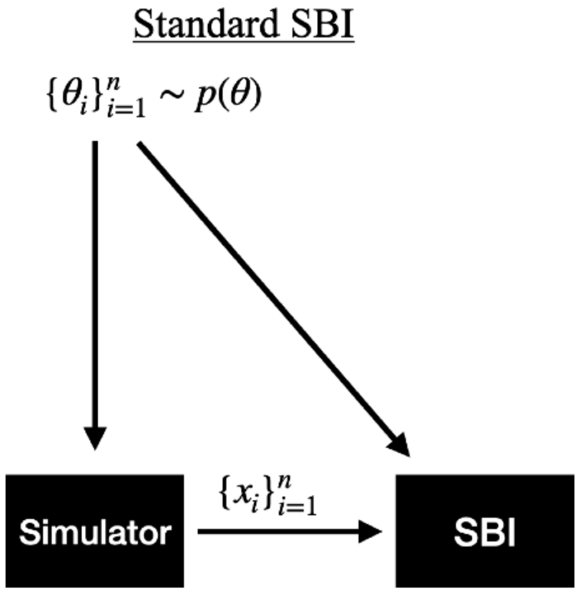
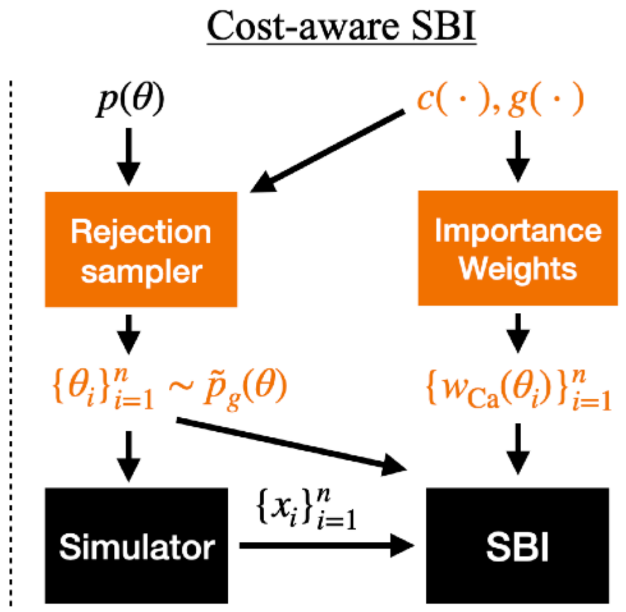
Putting it all together!

$$\ell_{\mathrm{NLE}}(\phi) = -\frac{1}{n}\sum_{i=1}^{n} \log q_{\phi}(\mathbf{x}_i \,|\, \theta_i), \qquad \theta_i \sim p(\theta), \mathbf{x}_i \sim p(\,\cdot\,|\,\theta)$$



Putting it all together!

$$\ell_{\mathrm{Ca-NLE}}(\phi) = -\frac{1}{n}\sum_{i=1}^{n} w_{\mathrm{Ca}}(\theta_i) \log q_\phi(x_i \mid \theta_i), \qquad \theta_i \sim \tilde{p}_g(\theta), x_i \sim p(\cdot \mid \theta)$$



Some reassuring results

Importance sampling can have infinite variance!!!

Some reassuring results

- Suppose that $g_{\max} = \sup_{\theta \in \Theta} g(c(\theta)) < \infty$. Then:

Some reassuring results

- Suppose that $g_{\max} = \sup_{\theta \in \Theta} g(c(\theta)) < \infty$. Then:
    1. The weights are bounded: $\dfrac{g_{\min}}{n g_{\max}} \leq w_{\text{Ca}}(\theta_i) \leq \dfrac{g_{\max}}{n g_{\min}} \qquad \forall i \in \{1, \ldots, n\},$

Some reassuring results

- Suppose that $g_{\max} = \sup_{\theta \in \Theta} g(c(\theta)) < \infty$. Then:

2. If f is square-integrable; i.e. $\int_\Theta f(\theta)^2 \pi(\theta) d\theta < \infty$, then Var$(\hat{\mu}_{\mathrm{Ca}}) = \sigma^2_{\mathrm{Ca}}$ where:

$$\frac{g_{\min}}{g_{\max}} \left(\sigma^2_{\mathrm{MC}} - \frac{\mu^2}{n} \right) \leq \sigma^2_{\mathrm{Ca}} \leq \frac{g_{\max}}{g_{\min}} \left(\sigma^2_{\mathrm{MC}} - \frac{\mu^2}{n} \right).$$

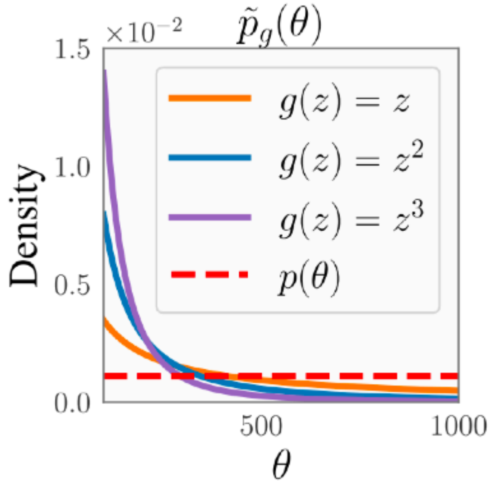
Some reassuring results

- Suppose that $g_{\max} = \sup_{\theta \in \Theta} g(c(\theta)) < \infty$. Then:

3. The ESS is bounded: $\left(\frac{g_{\min}}{g_{\max}}\right)^2 \leq \text{ESS} \leq \left(\frac{g_{\max}}{g_{\min}}\right)^2$.

A Gamma simulator

- $\mathbb{P}_\theta =$ Gamma$(\theta, 1)$,
- Simulator: Ahrens-Dieter acceptance-rejection method.
- Method: ABC!



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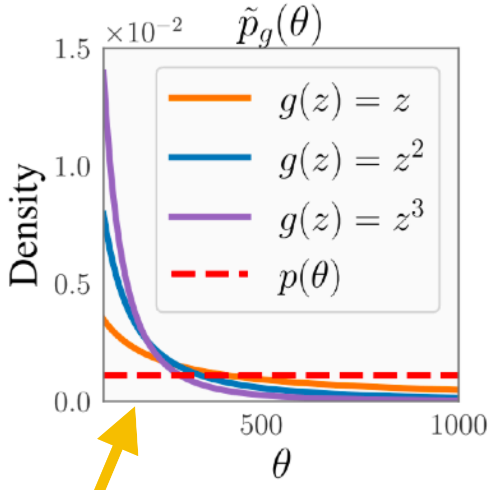

Cost-aware pushes us to sample from small θ values!

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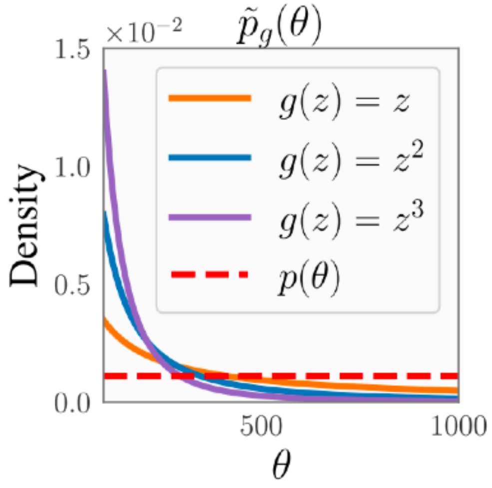

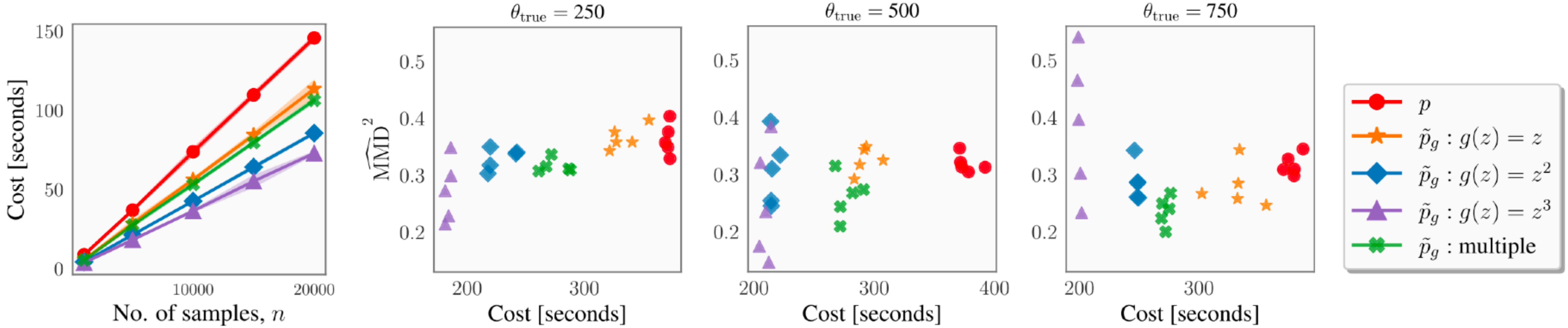

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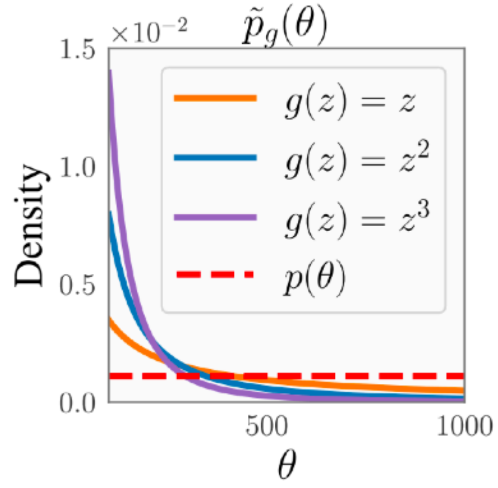

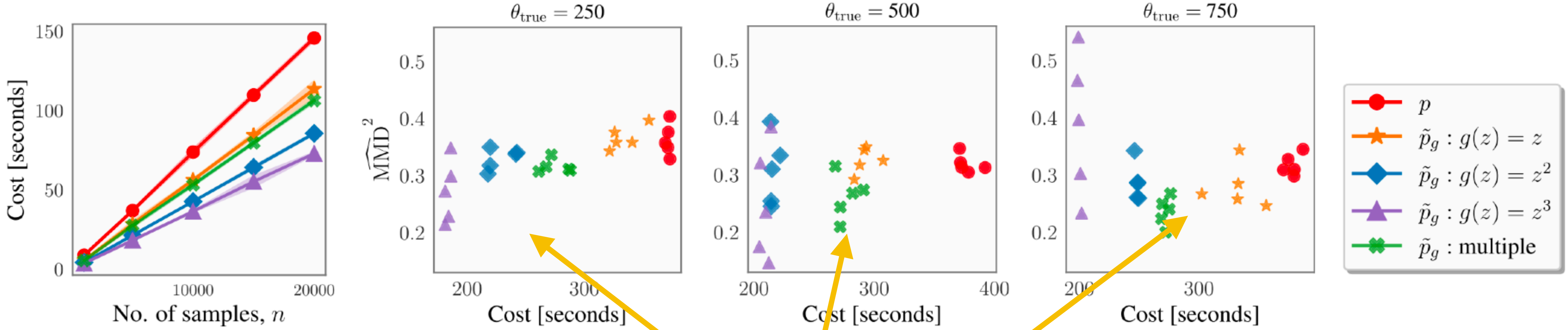

Being cost-aware tends to reduce your cost without a loss of accuracy!

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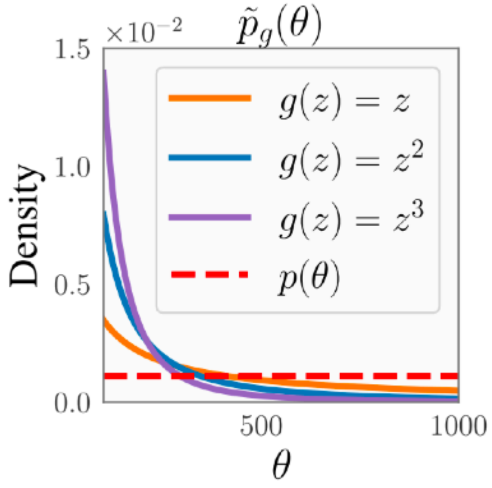
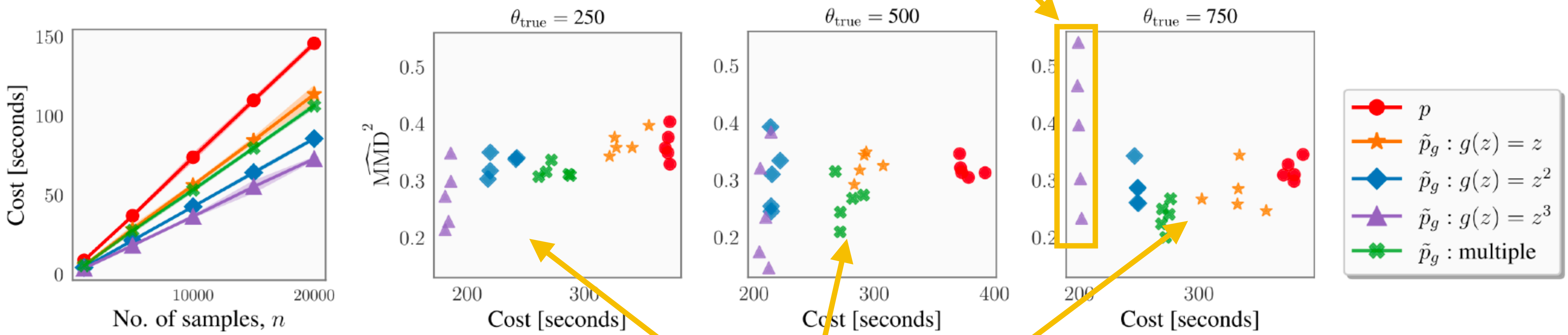


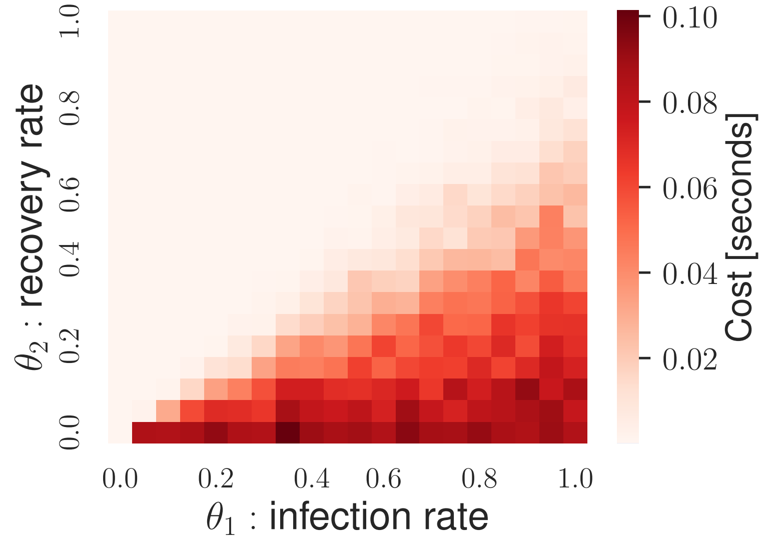
If truth in expensive region, being 'too' cost-aware won't be great!



Being cost-aware tends to reduce your cost without a loss of accuracy!

Some epidemiological models

- We consider three different models with 1,2 and 3 parameters respectively, and use NPE.



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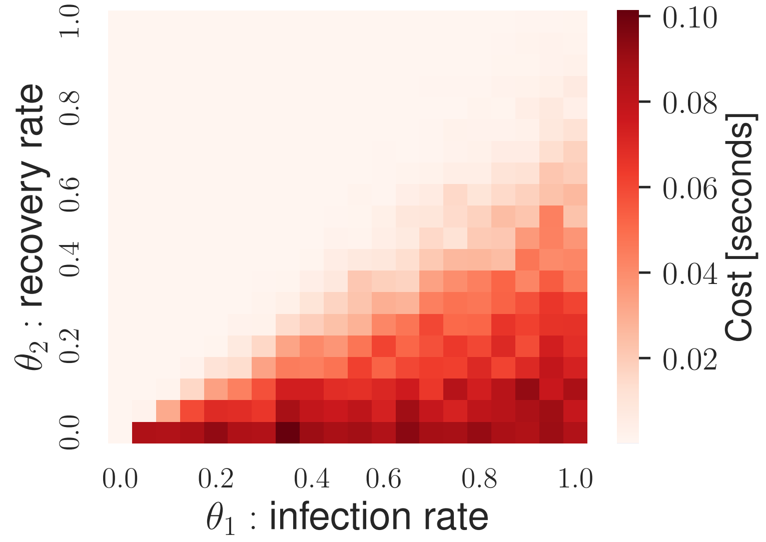

	$\widehat{\mathrm{MMD}}^2$ (↓)					Time saved (↑)			
	NPE	Ca-NPE $g(z)=z^{0.5}$	Ca-NPE $g(z)=z$	Ca-NPE $g(z)=z^2$	Ca-NPE multiple	Ca-NPE $g(z)=z^{0.5}$	Ca-NPE $g(z)=z$	Ca-NPE $g(z)=z^2$	Ca-NPE multiple
Homogen.	0.02(0.02)	0.02(0.01)	0.02(0.02)	0.23(0.08)	0.05(0.04)	16%(2)	38%(2)	70%(2)	30%(5)
Temporal	0.03(0.03)	0.06(0.03)	0.07(0.03)	0.07(0.03)	0.05(0.04)	36%(4)	65%(2)	85%(1)	24%(5)
Bernoulli	0.02(0.00)	0.02(0.00)	0.02(0.01)	0.04(0.01)	0.02(0.00)	23%(4)	37%(4)	47%(3)	25%(6)

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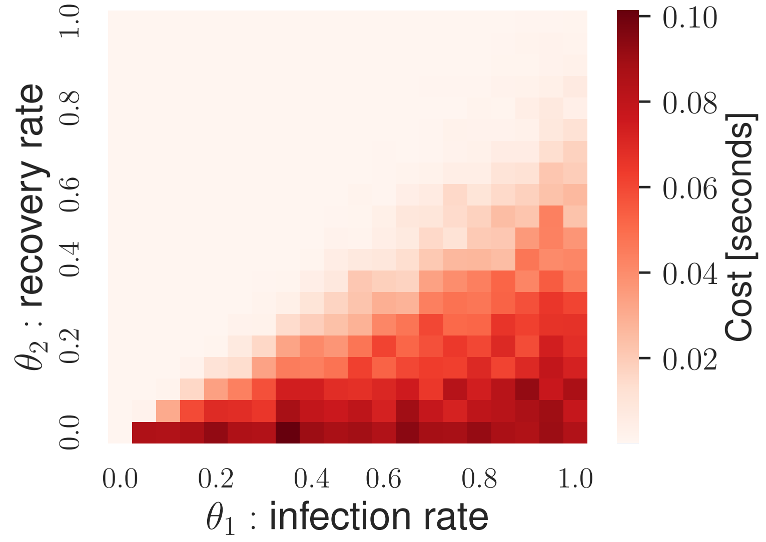

$g(z) = z^{0.5}$: Same accuracy but modest improvement!

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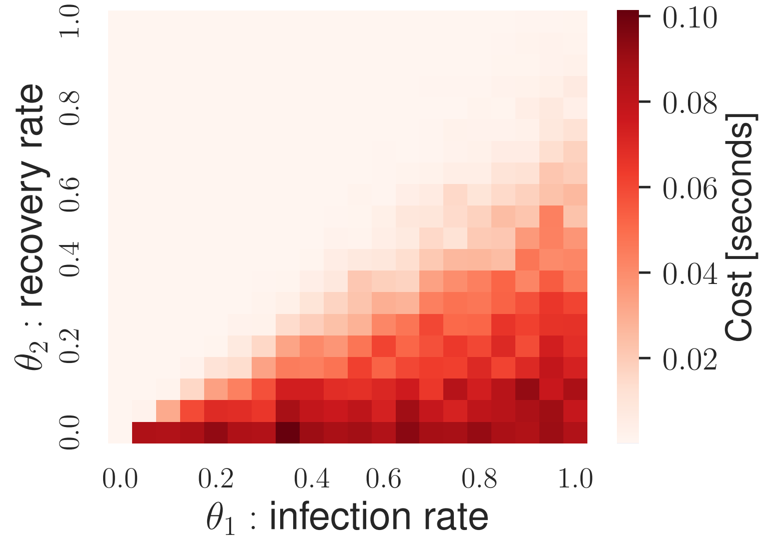

$g(z) = z$: Still same accuracy but slightly better improvement!

	$\widehat{\mathrm{MMD}}^2$ (↓)					Time saved (↑)			
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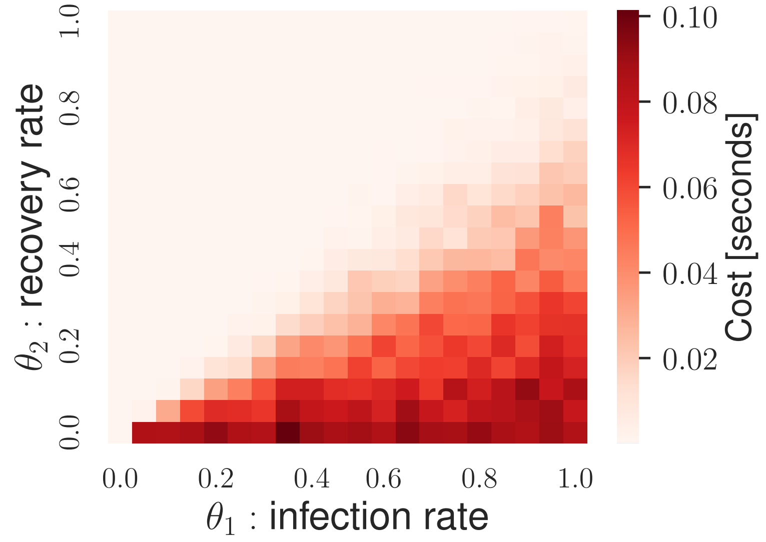
$g(z) = z^2$: Worse accuracy but much cheaper



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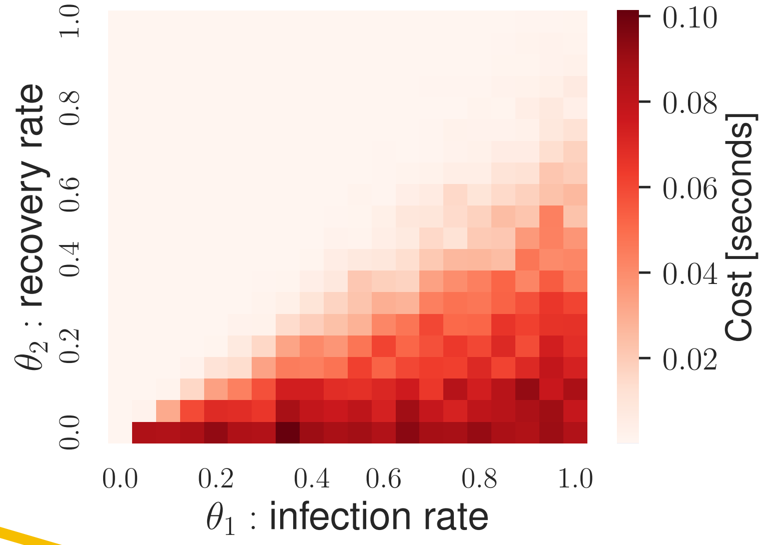

Typically slight loss of accuracy but decent reduction in cost!

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This is typically very cheap, and simulations can be re-used for inference!

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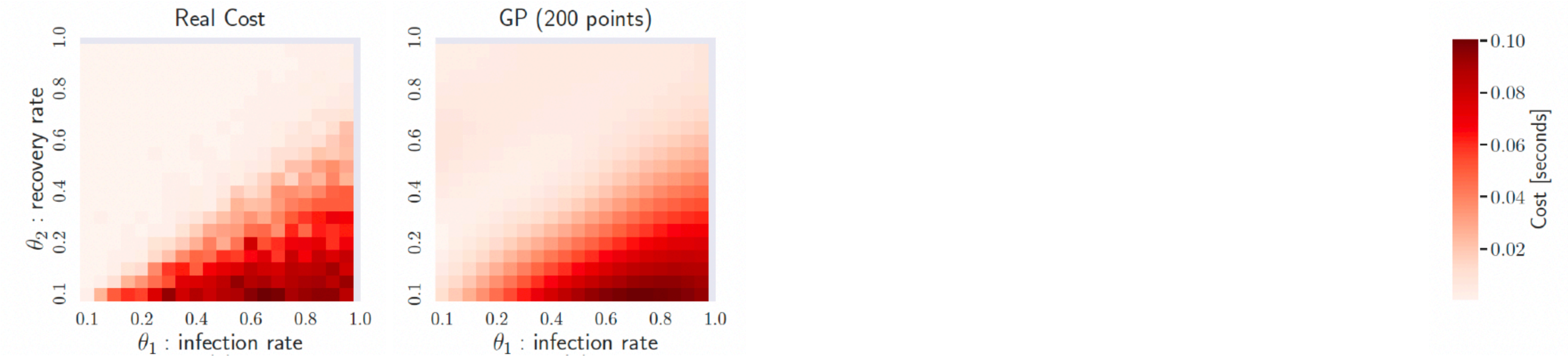

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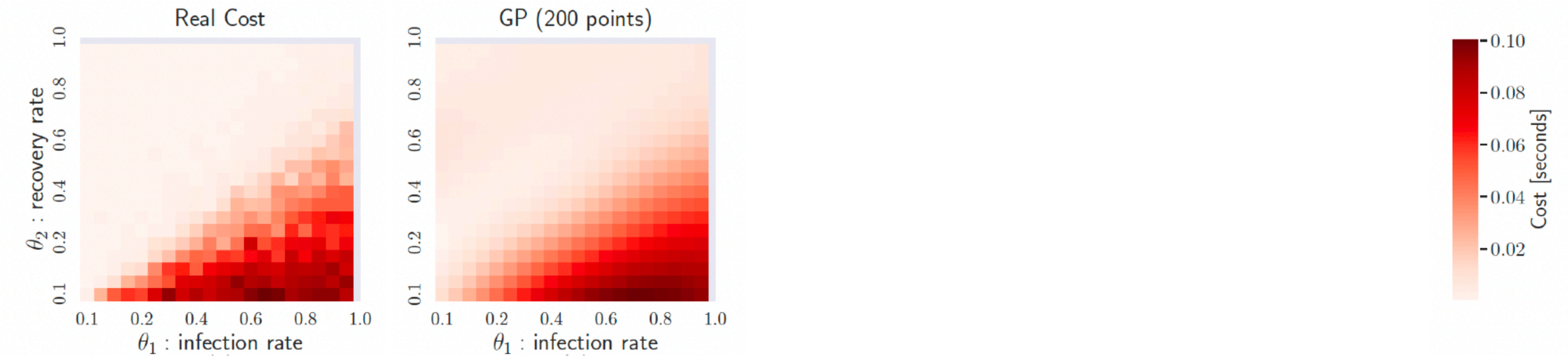

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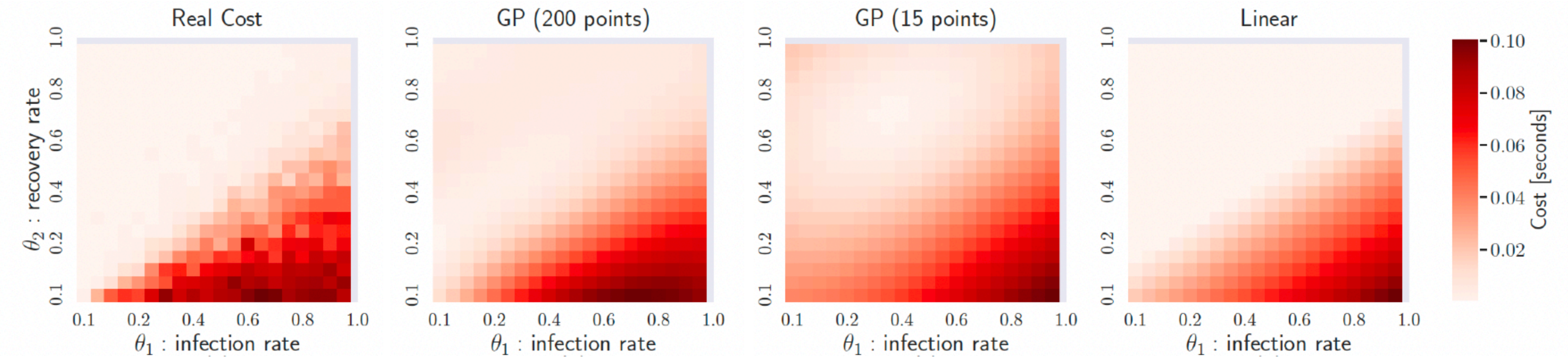

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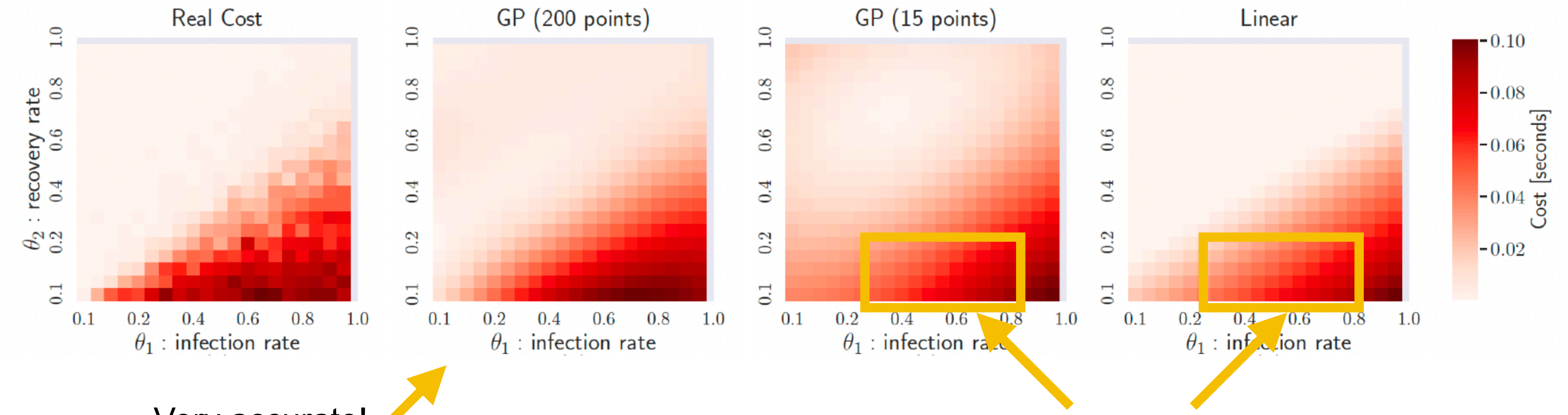
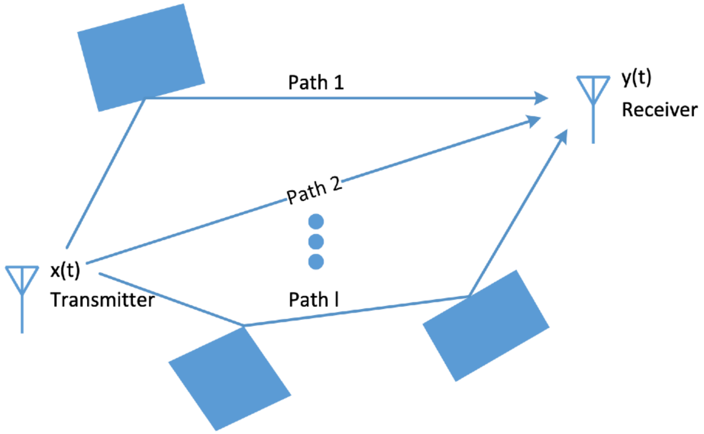


Very accurate!

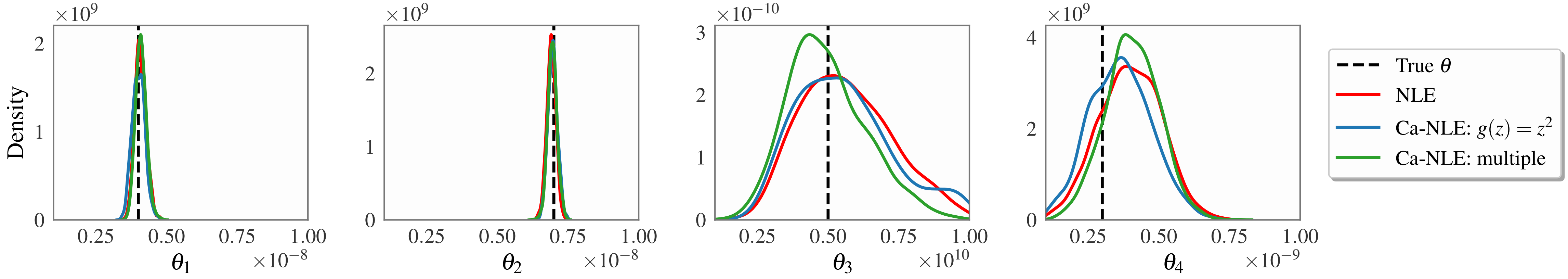
Clearly not perfect, but still pretty good…

Back to radio-propagation



<u>Computational Cost</u>

- **Standard NLE:** 15.6h,
- **Cost-aware NLE:** 8.8h!!



Bharti, A., **Briol, F-X.**, Pedersen, T. (2022). A general method for calibrating stochastic radio channel models with kernels. IEEE Transactions on Antennas and Propagation, vol. 70, no. 6, pp. 3986-4001, June 2022.

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Conclusion

- We proposed a novel importance sampling algorithm which focuses on **down weighting sampling** in regions with a **large downstream cost.**
- Although I presented this for NLE/NPE, we also have experiments for ABC and it could be applied to any other sampling-based SBI method.
- Need more computational statisticians engaging with neural-based simulation inference!

Any Questions?

Paper: Bharti, A., Huang, D., Kaski, S., & Briol, F.-X. (2025). Cost-aware simulation-based inference. International Conference on Artificial Intelligence and Statistics, 28–36.

Code: https://github.com/huangdaolang/cost-aware-sbi

Robust Bayesian simulation-based inference

Paper: Dellaporta, C., Knoblauch, J., Damoulas, T. & **Briol, F-X** (2022). Robust Bayesian inference for simulator-based models via the MMD posterior bootstrap. AISTATS, 943-970. Best paper award.

Code: https://github.com/haritadell/npl_mmd_project

Connections with Jeremias' course

Optimisation-centric posteriors / Generalised Variational Inference

$$q_n^*(\theta) = \arg\min_{q \in \mathcal{Q}} \left\{ \mathscr{L}(q, x_{1:n}) \quad + \quad \mathrm{D}(q, \pi) \right\}$$

Gibbs/Generalised/

$$\pi_n^{\mathsf{L}}(\theta \mid x_{1:n}) = \frac{\exp\{-\mathsf{L}(x_{1:n}, p_\theta)\} \cdot \pi(\theta)}{\int \exp\{-\mathsf{L}(x_{1:n}, p_\theta)\} \cdot \pi(\theta) d\theta}$$

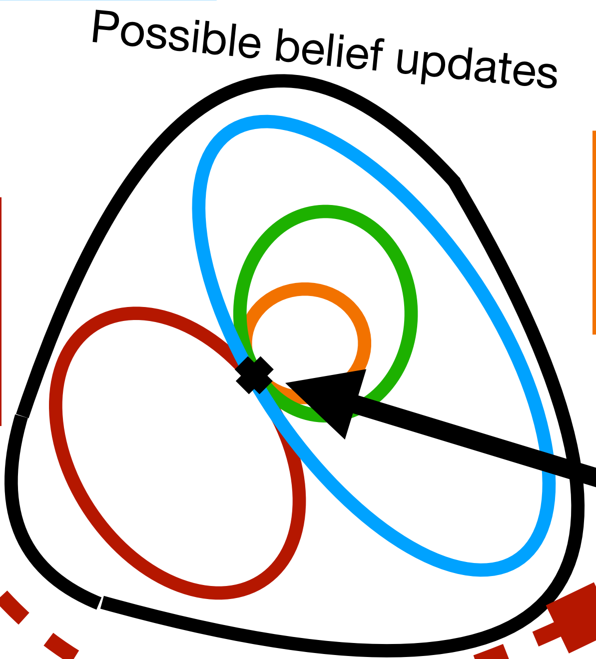
Martingale posteriors & resampling-based approaches

For $i = 1,2,\ldots$

$$X_{n+i+1} \sim p(X_{n+i} \mid x_{1:n}, X_{n+1:n+i})$$

$$\theta^\infty = \mathrm{argmin}_{\theta \in \Theta} \mathsf{L}\left([x_{1:n}, X_{n+1:\infty}], \theta\right)$$

[See Fong, Holmes, & Walker (2023)]



Power/Fractional/

$$\pi_n^{(\lambda)}(\theta \mid x_{1:n}) = \frac{p(x_{1:n} \mid \theta)^\lambda \cdot \pi(\theta)}{\int p(x_{1:n} \mid \theta)^\lambda \cdot \pi(\theta) d\theta}$$

Bayes'

$$\pi_n(\theta \mid x_{1:n}) = \frac{p(x_{1:n} \mid \theta) \cdot \pi(\theta)}{\int p(x_{1:n} \mid \theta) \cdot \pi(\theta) d\theta}$$

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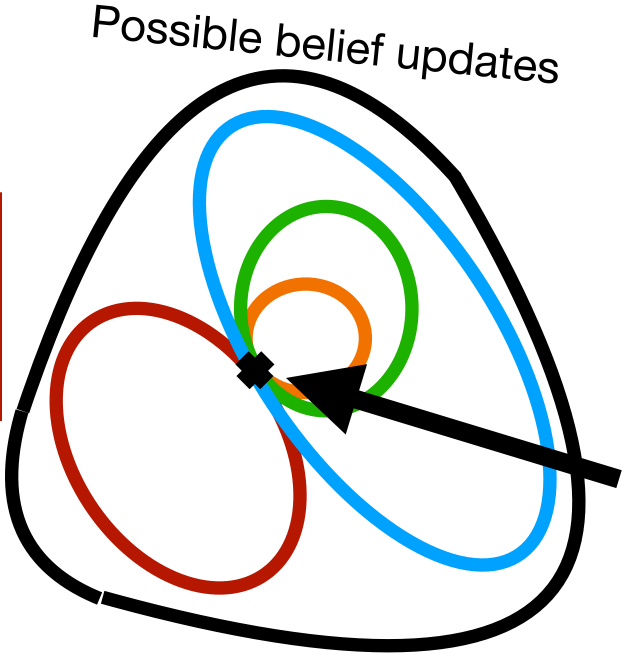

Non-parametric Learning

- Place a Dirichlet process $\mathrm{DP}(\alpha; \mathbb{F})$ prior on $\mathbb{Q}$

Lyddon, S., Walker, S., & Holmes, C. (2018). Nonparametric learning from Bayesian models with randomized objective functions. *NeurIPS*, 2071–2081.

Fong, E., Lyddon, S., & Holmes, C. (2019). Scalable nonparametric sampling from multimodal posteriors with the posterior bootstrap. ICML, 3443–3464.

Non-parametric Learning

- Place a Dirichlet process $\mathrm{DP}(\alpha; \mathbb{F})$ prior on $\mathbb{Q}$

- Condition this prior on the observed data $y_1, \ldots, y_n \sim \mathbb{Q}$ to get a posterior:

$$\mathrm{DP}(\alpha'; \mathbb{F}') \qquad \alpha' = \alpha + n \qquad \mathbb{F}' = \frac{\alpha}{\alpha + n}\mathbb{F} + \frac{n}{\alpha + n}\mathbb{Q}_n$$

Lyddon, S., Walker, S., & Holmes, C. (2018). Nonparametric learning from Bayesian models with randomized objective functions. *NeurIPS*, 2071–2081.

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Non-parametric Learning

Rather than doing inference on $\{\mathbb{P}_\theta\}_{\theta\in\Theta}$ (which could be misspecified), we do inference on $\mathbb{Q}$!

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We still care about $\{\mathbb{P}_\theta\}_{\theta\in\Theta}$, so we map back to parameter space!

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The posterior bootstrap (i.e. NPL in practice)

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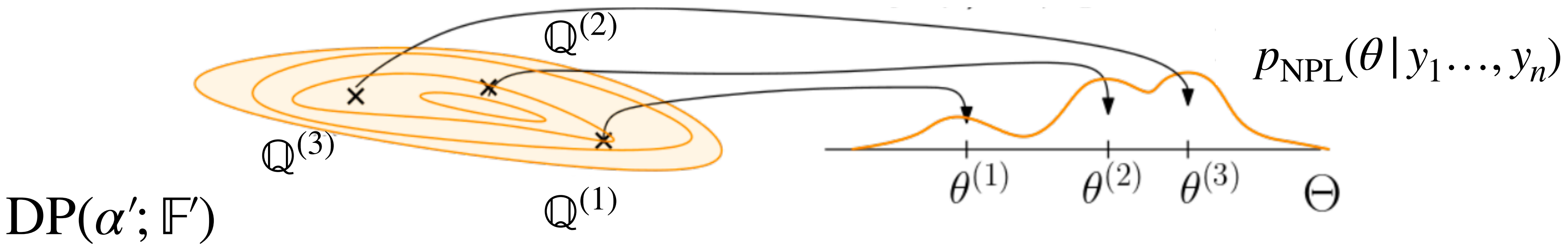

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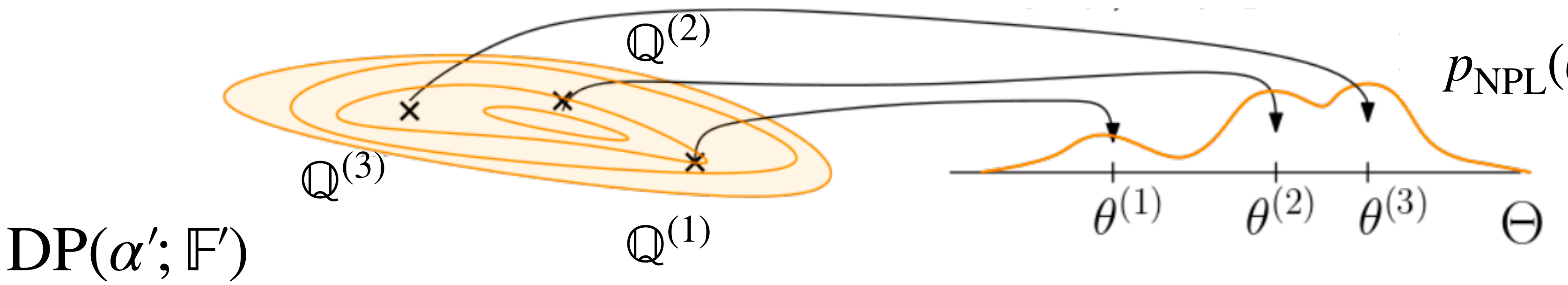

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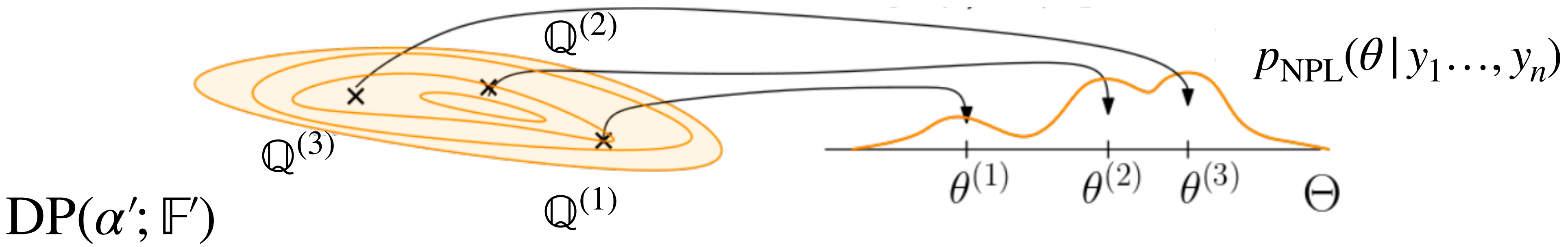

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Bounded!

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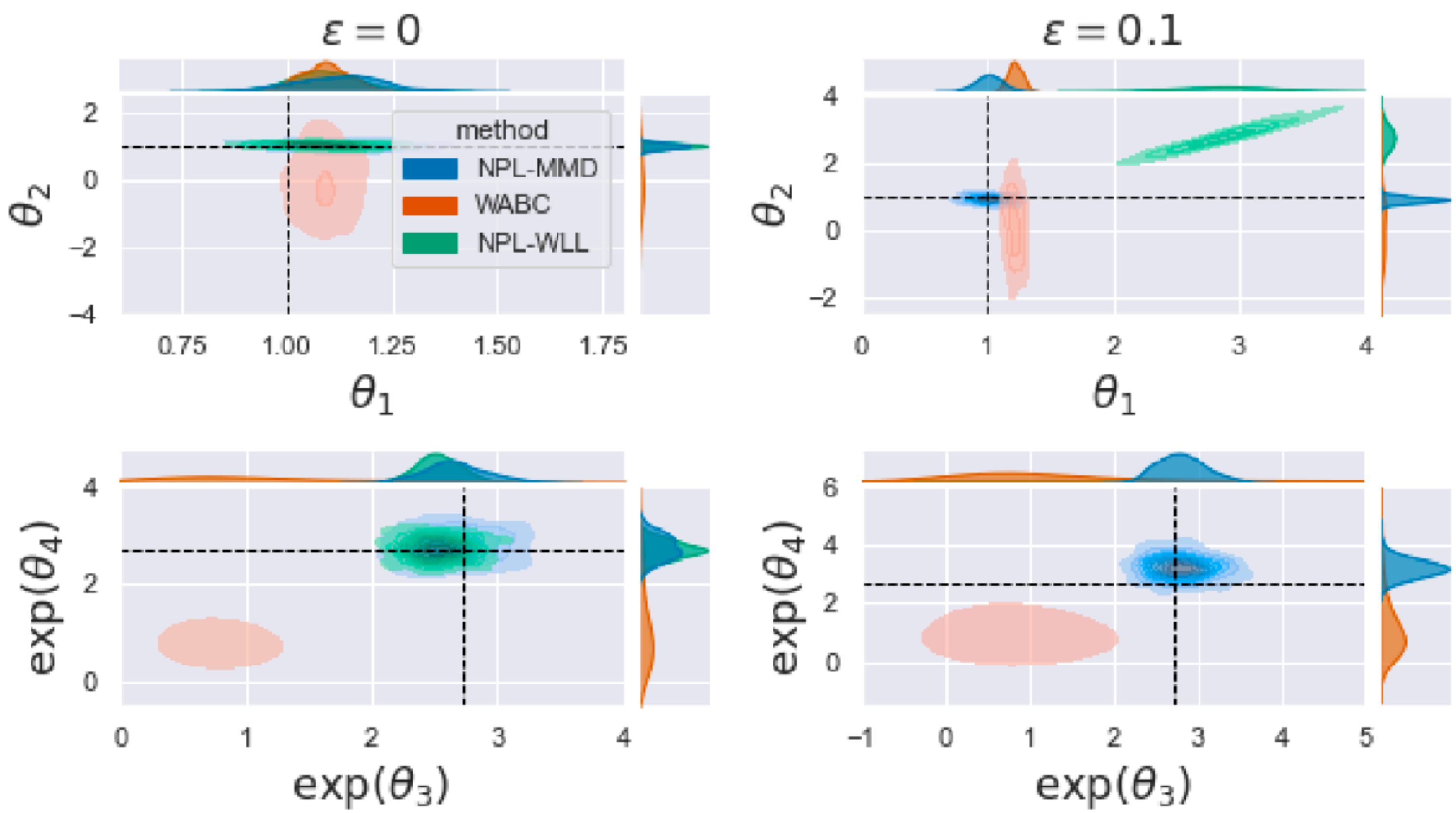
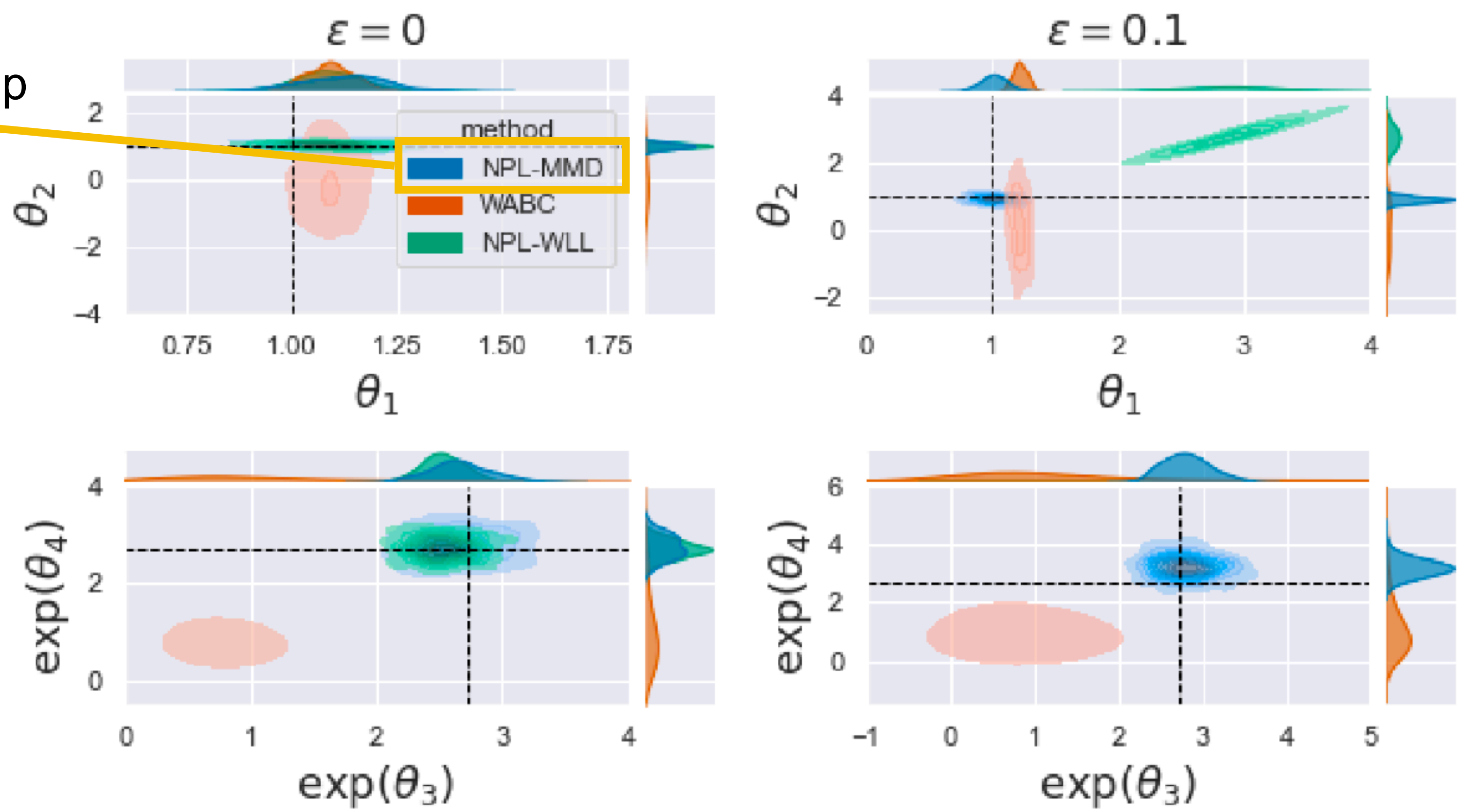
→ **Double robustness** robust inference procedure and robust estimator!

Example 1: Misspecified Gaussian



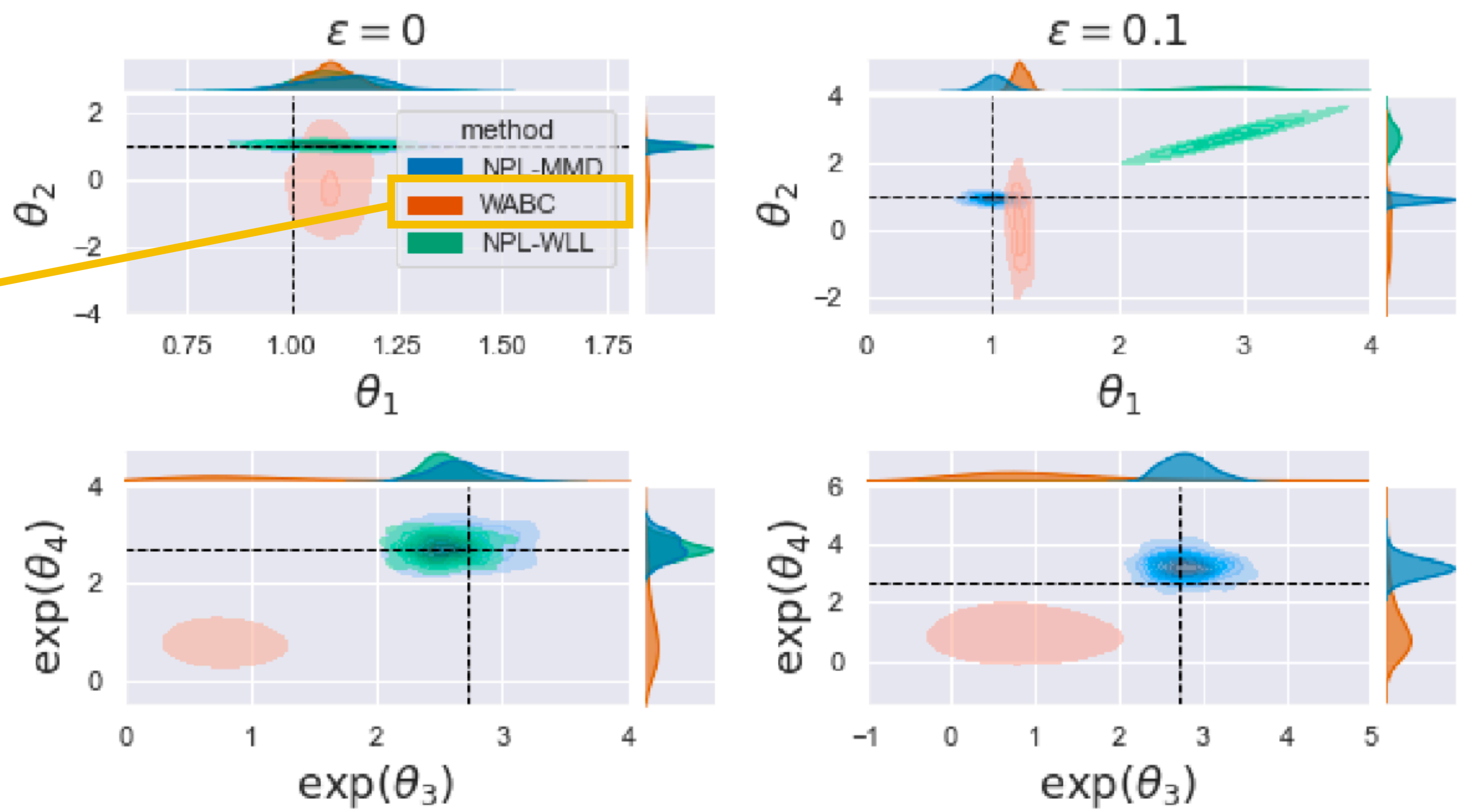
Example 1: Misspecified Gaussian

NPL-MMD:
MMD posterior bootstrap



Example 1: Misspecified Gaussian

WABC:
ABC with Wasserstein distance



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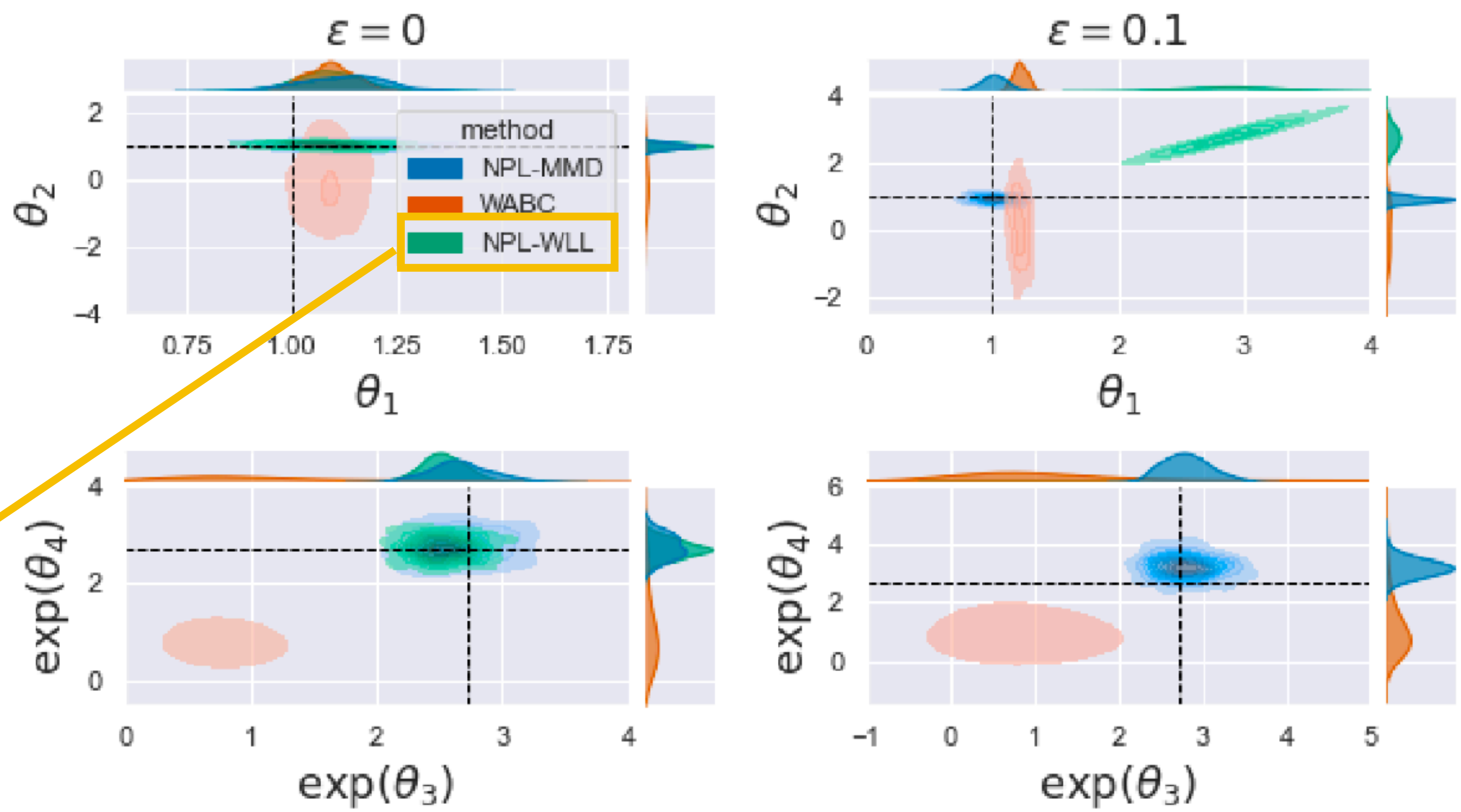


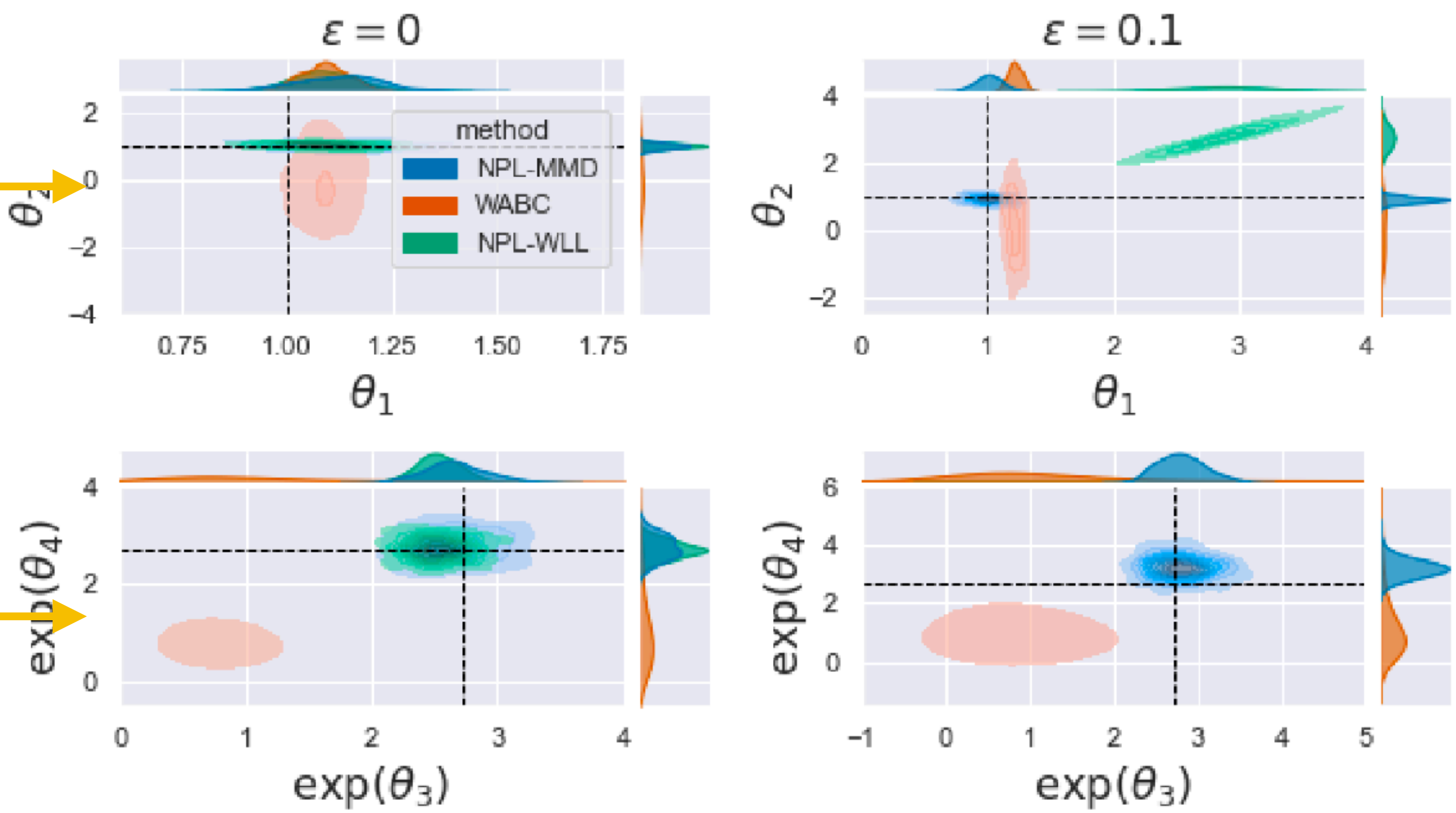
NPL-WLL:

$$l(x, \theta) = -\beta \log p(x \mid \theta)$$

Example 1: Misspecified Gaussian

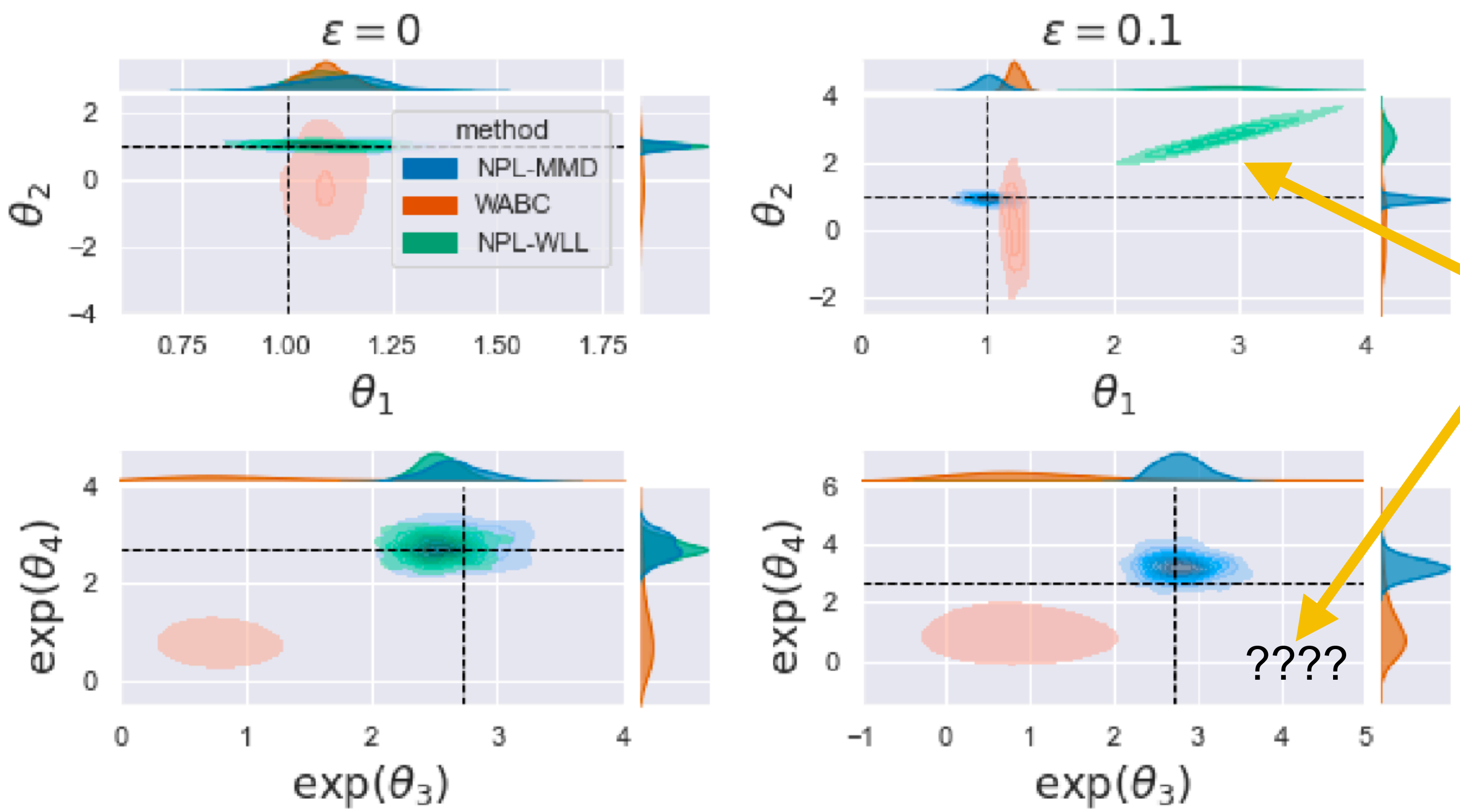
'Easy' parameters;
All do ok!

'Hard' parameters;
WABC already struggles a bit



Well-specified case!

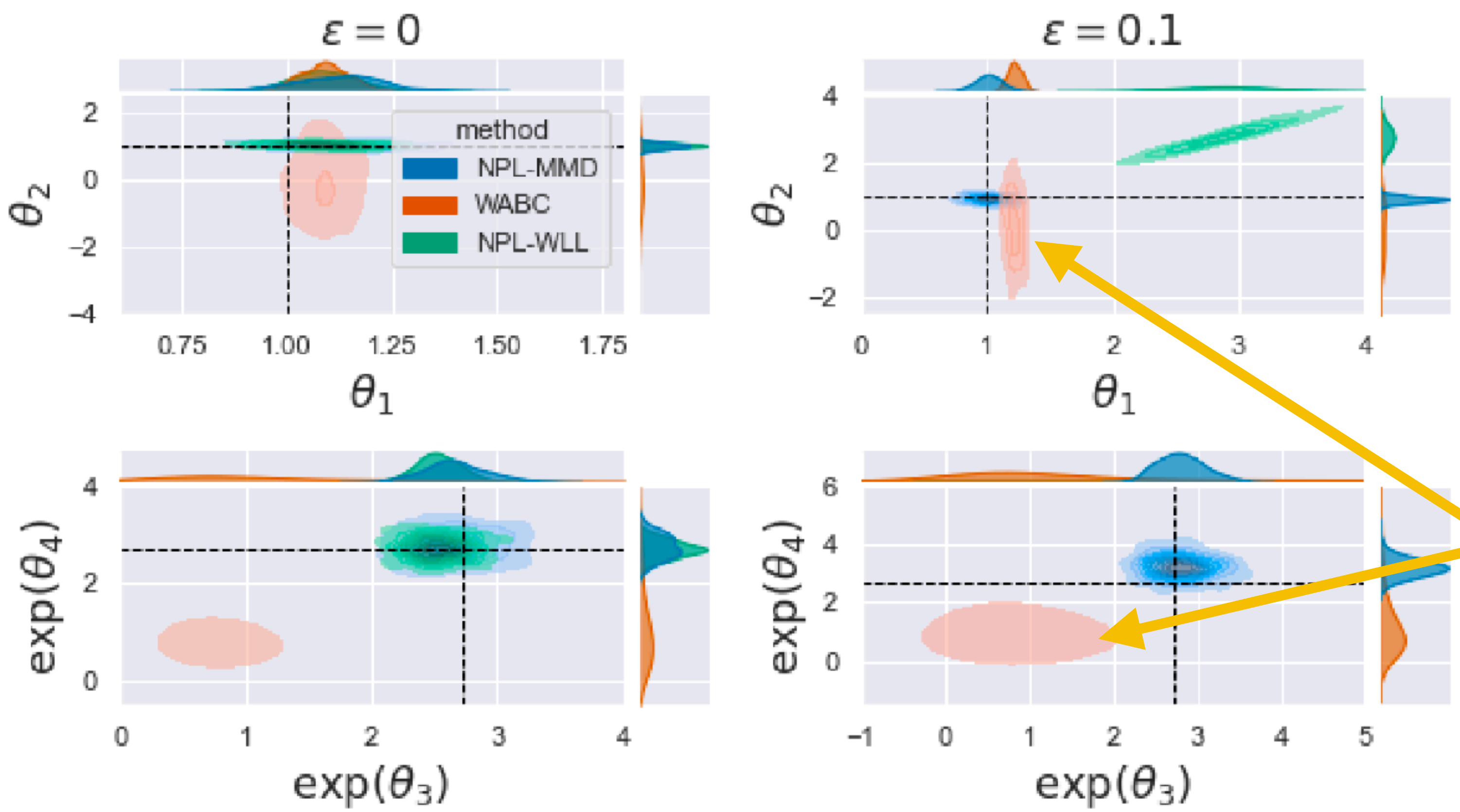
Example 1: Misspecified Gaussian



Misspecified case

NPL-WLL really struggles

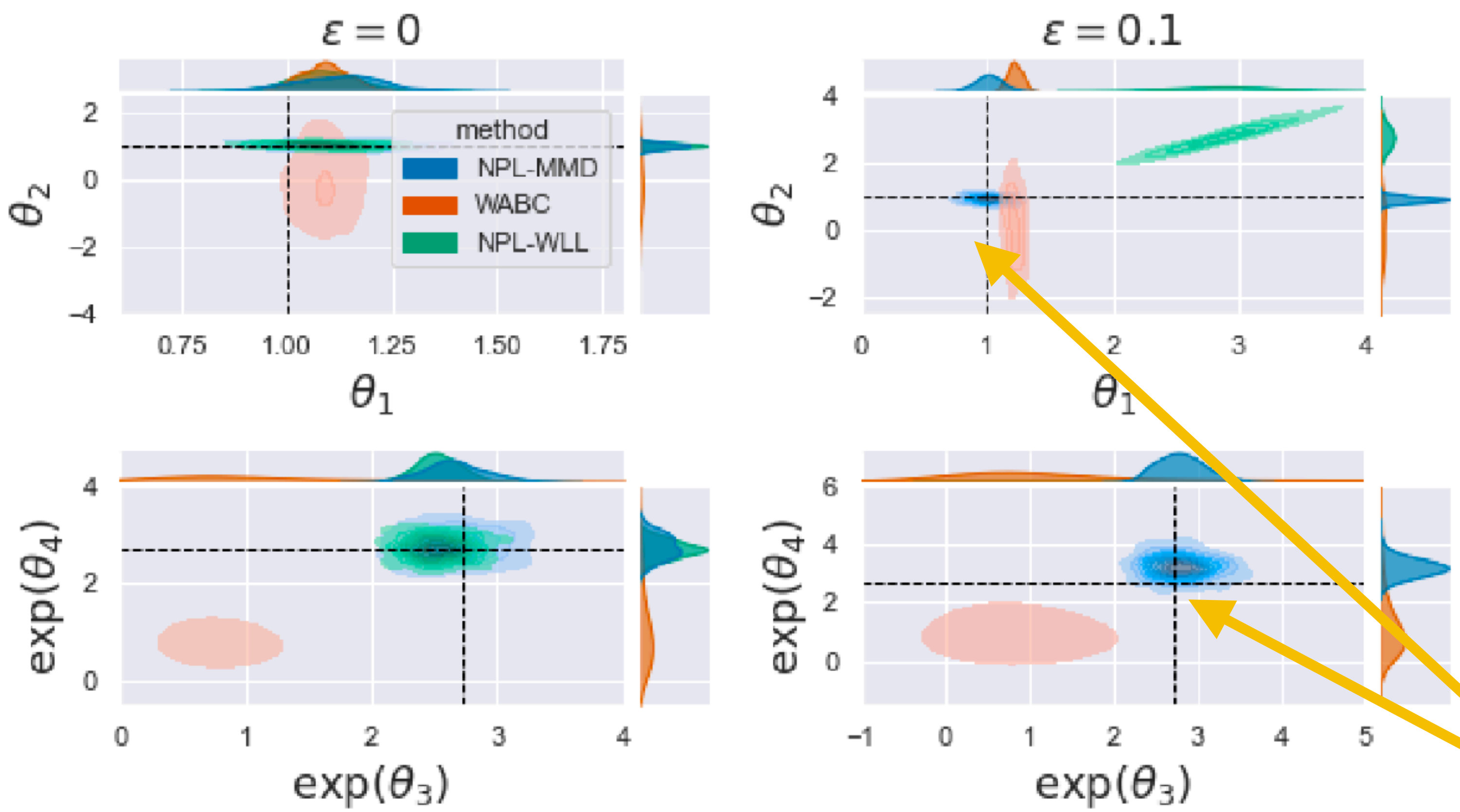
Example 1: Misspecified Gaussian



Misspecified case

WABC impacted!

Example 1: Misspecified Gaussian



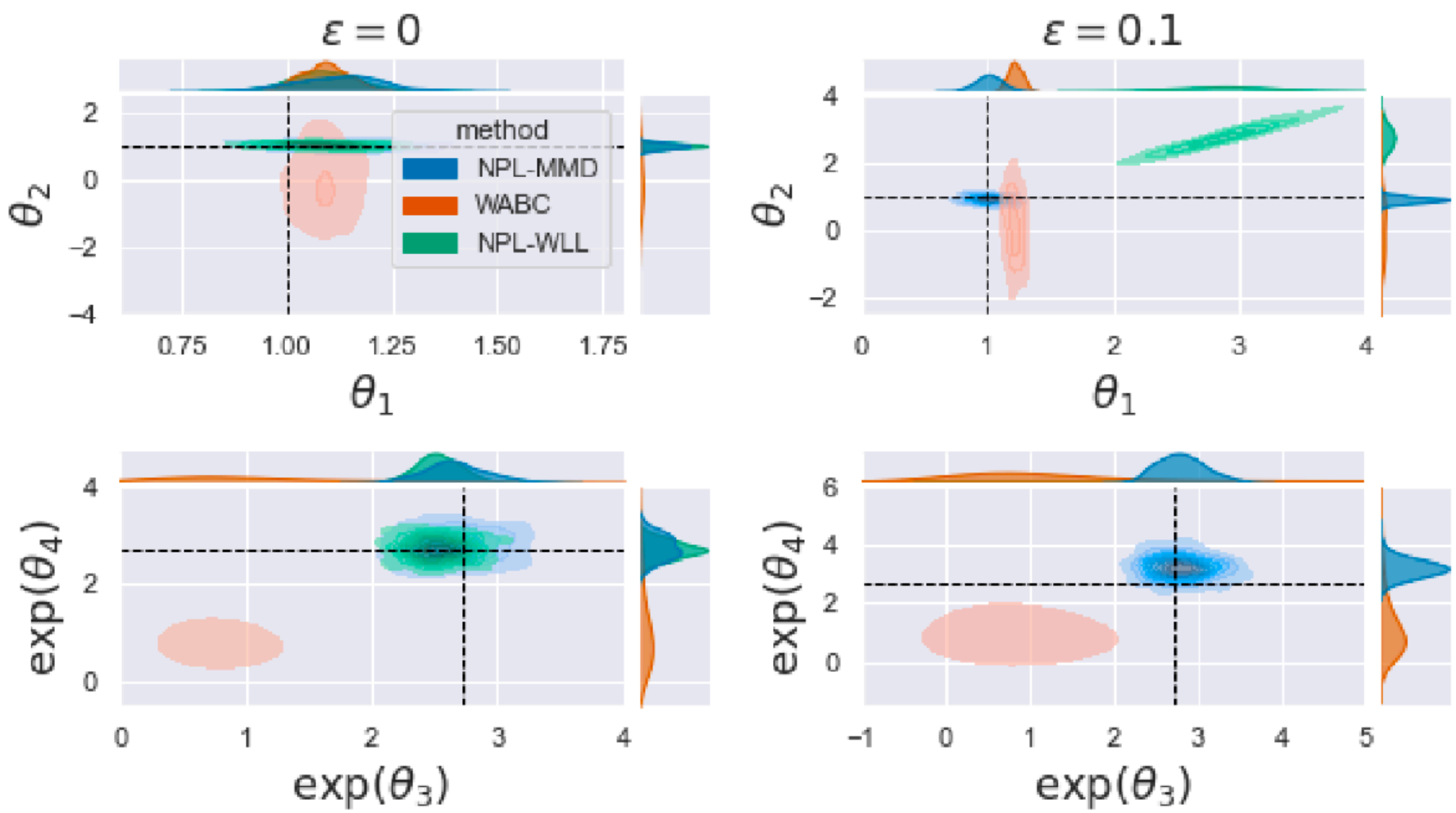
Misspecified case

MMD posterior bootstrap barely impacted!!

Example 1: Misspecified Gaussian

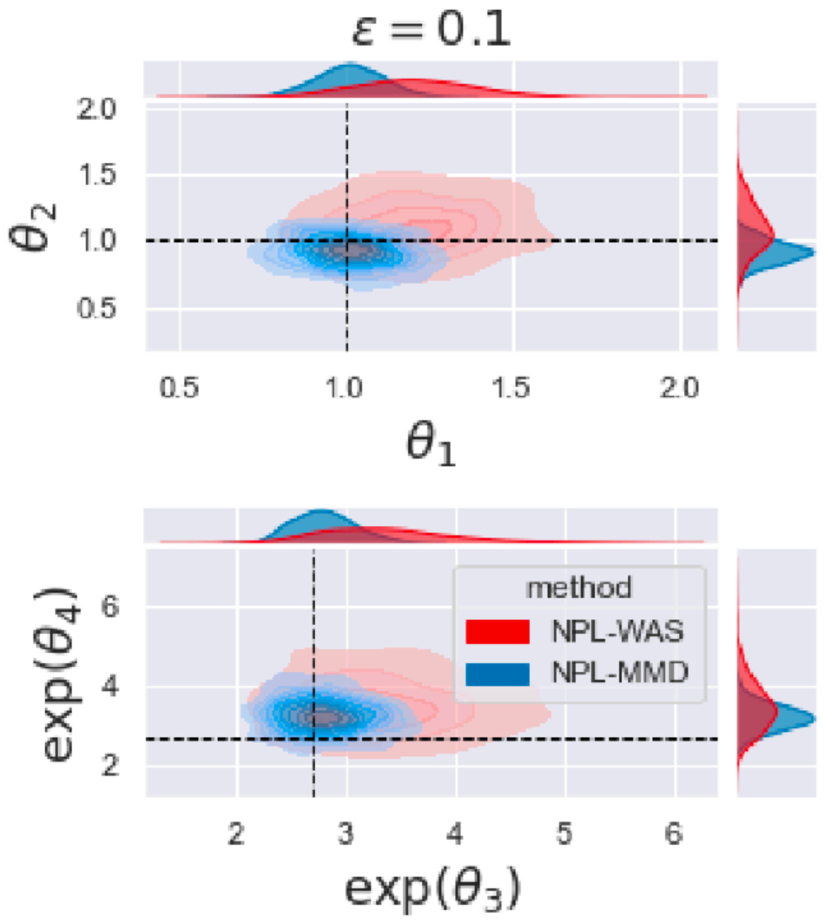
Time to run:
NPL-MMD: ≈ 2 mins
WABC: ≈ 1 hour

Misspecified case



Example 1 continued: Wasserstein NPL

- In principle, nothing stops us from using the Wasserstein instead of MMD.



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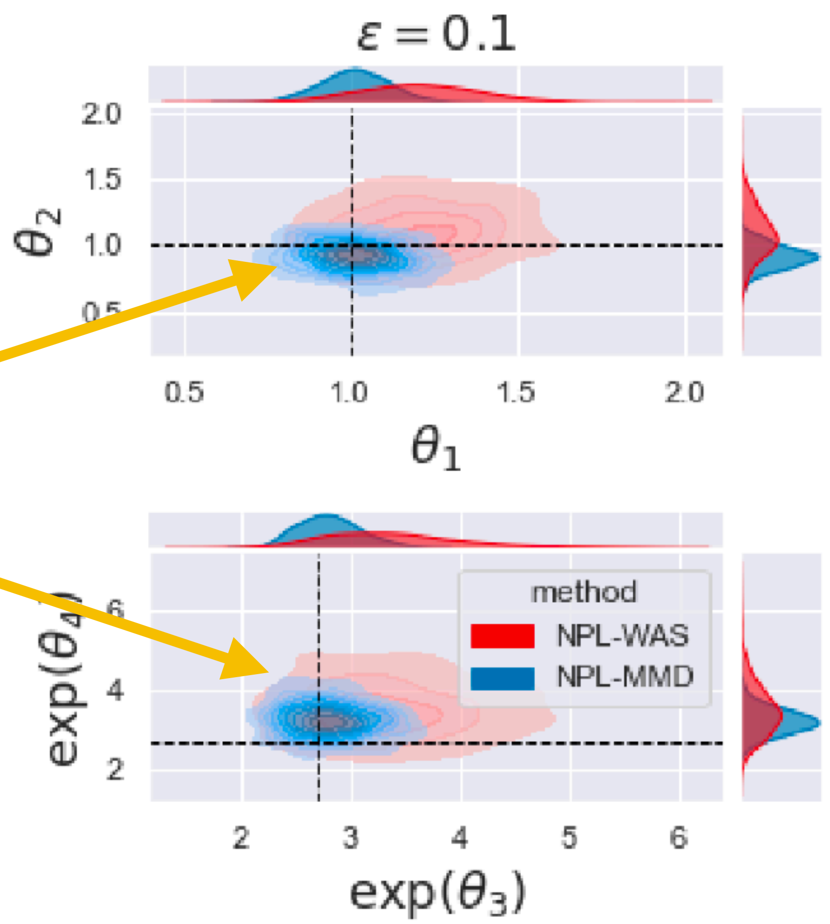
- In principle, nothing stops us from using the Wasserstein instead of MMD.
- The results are still reasonable thanks to NPL framework, but much more diffuse



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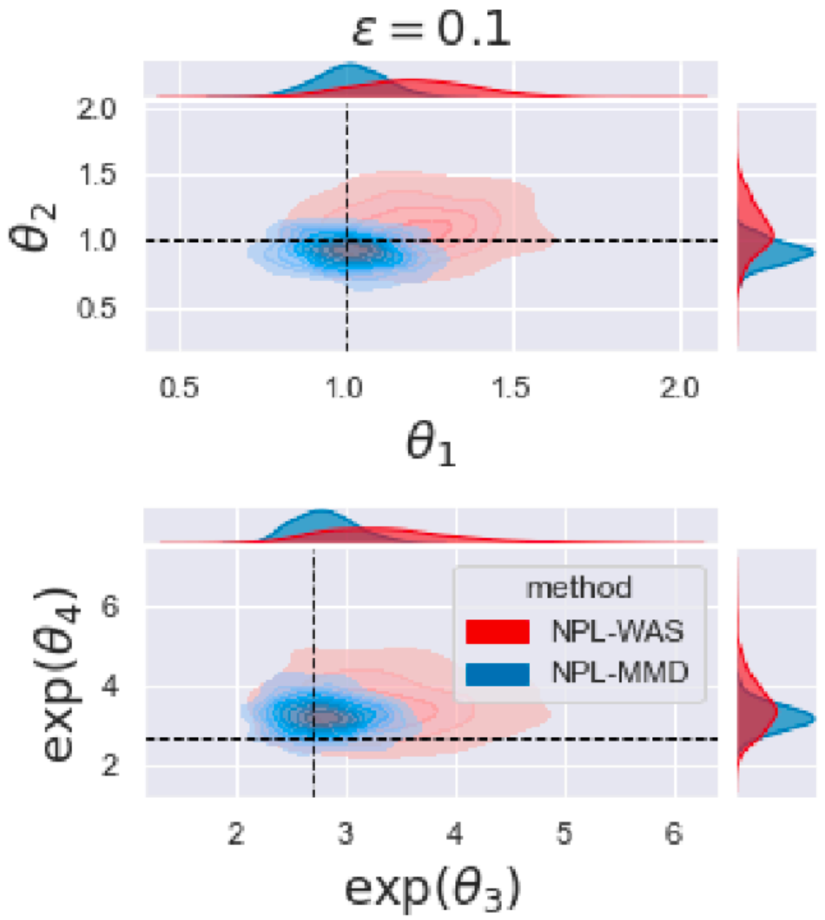
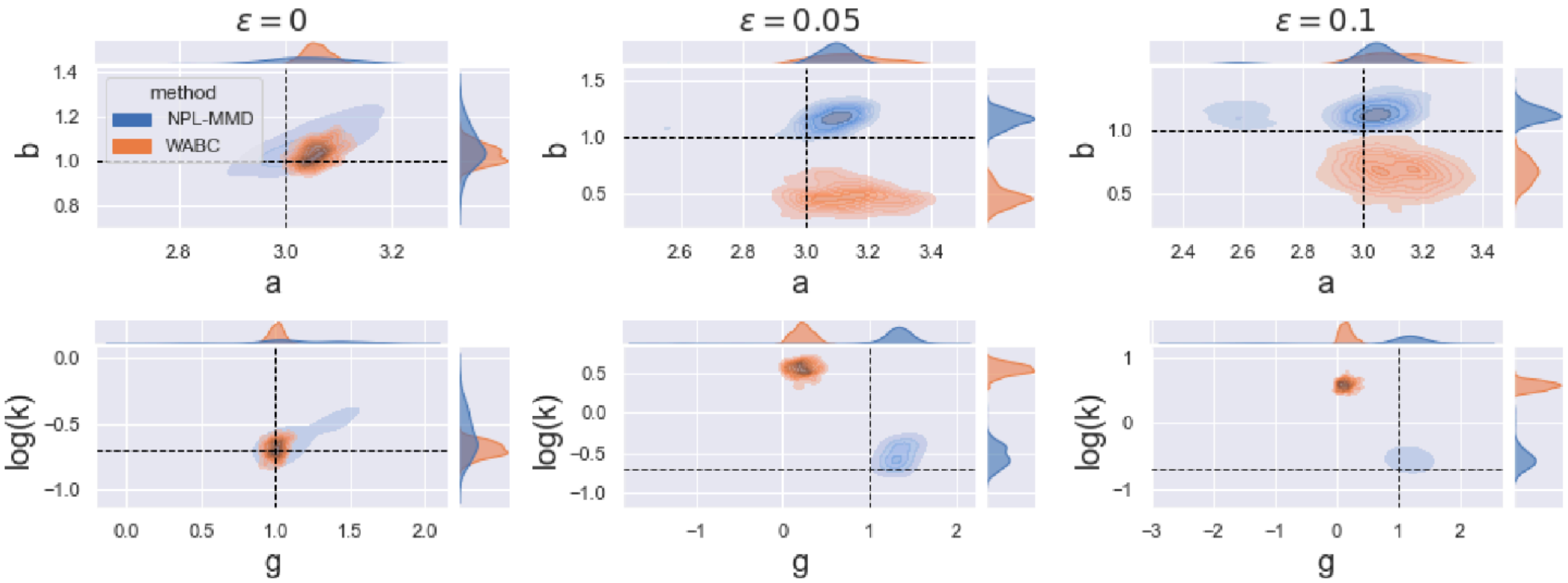


We really do gain from having both a **robust inference** framework **AND** a **robust estimator**…

Misspecified g-and-k distribution



$$G_\theta(u) = \theta_1 + \theta_2 \left[1 + 0.8 \left(\frac{1 - \exp(-\theta_3 z(u))}{1 + \exp(-\theta_3 z(u))} \right) \right] \left(1 + z(u)^2\right)^{\log(\theta_4)} z(u),$$

$$z(u) = \Phi^{-1}(u) = \sqrt{2}\,\mathrm{erf}^{-1}(2u - 1), \qquad u \sim \text{Unif}([0,1]),$$

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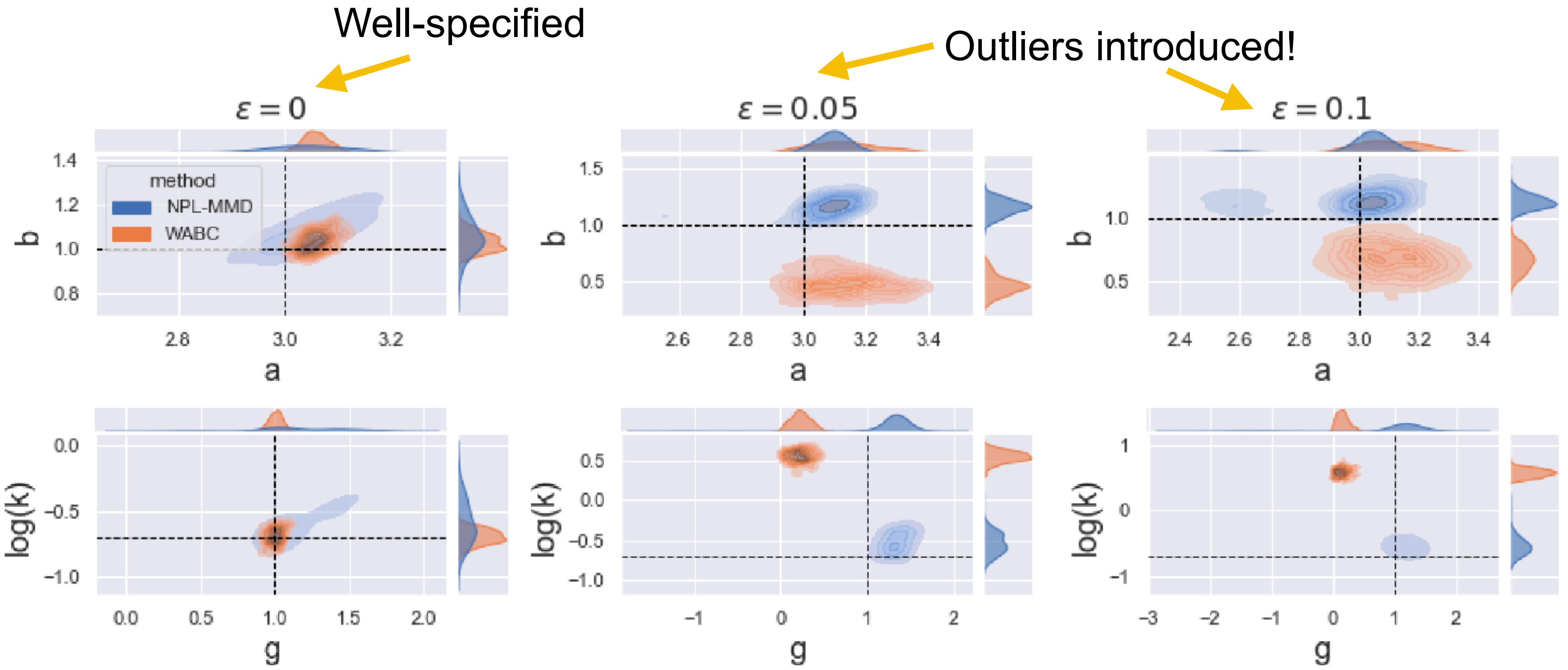

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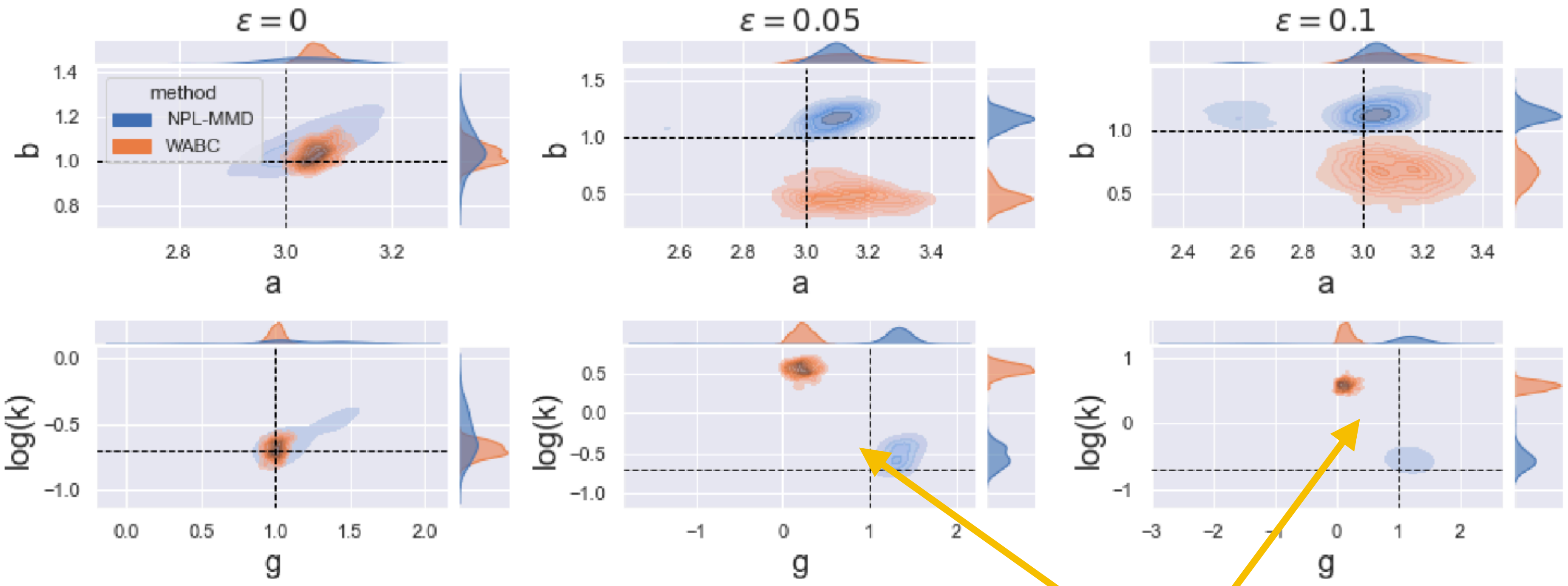

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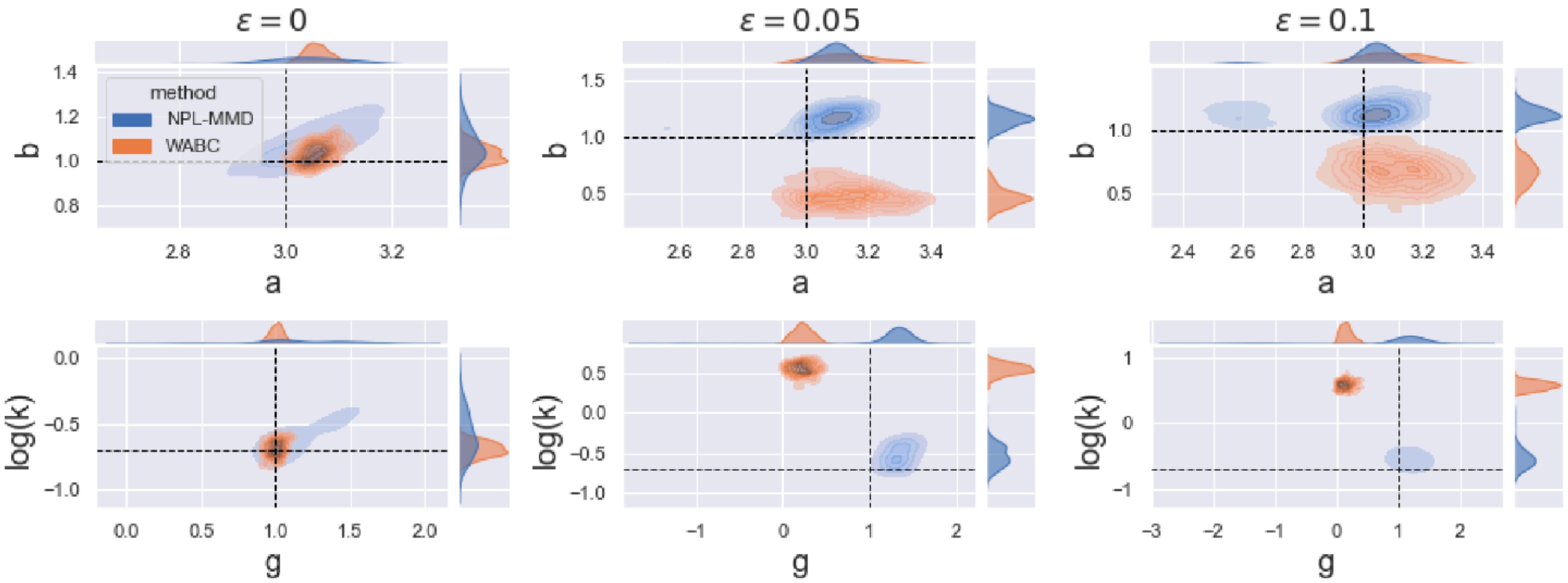
Wasserstein ABC really struggles with outliers, but the MMD posterior bootstrap is not significantly impacted

Misspecified g-and-k distribution

Time to run:
NPL-MMD: ≈ 30 sec
WABC: ≈ 100 sec



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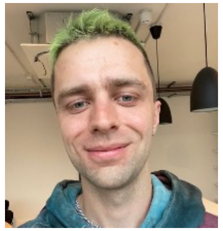
Going beyond iid…

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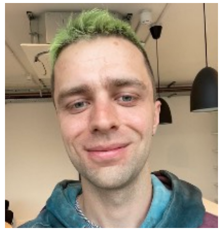
$$(\mathbb{P}_\theta)_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{x_i}, \qquad x_i = G_\theta(u_i), \qquad u_i \sim \text{Unif}[0,1]$$

Going beyond iid…

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- Can do better with:

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Niu, Z., Meier, J., & **Briol, F.-X.** (2023). Discrepancy-based inference for intractable generative models using quasi-Monte Carlo. Electronic Journal of Statistics, 17(1), 1411–1456.

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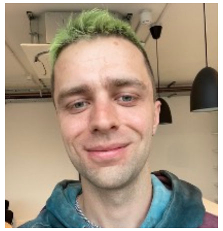
Going beyond iid…

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Non-equal weights

Grids

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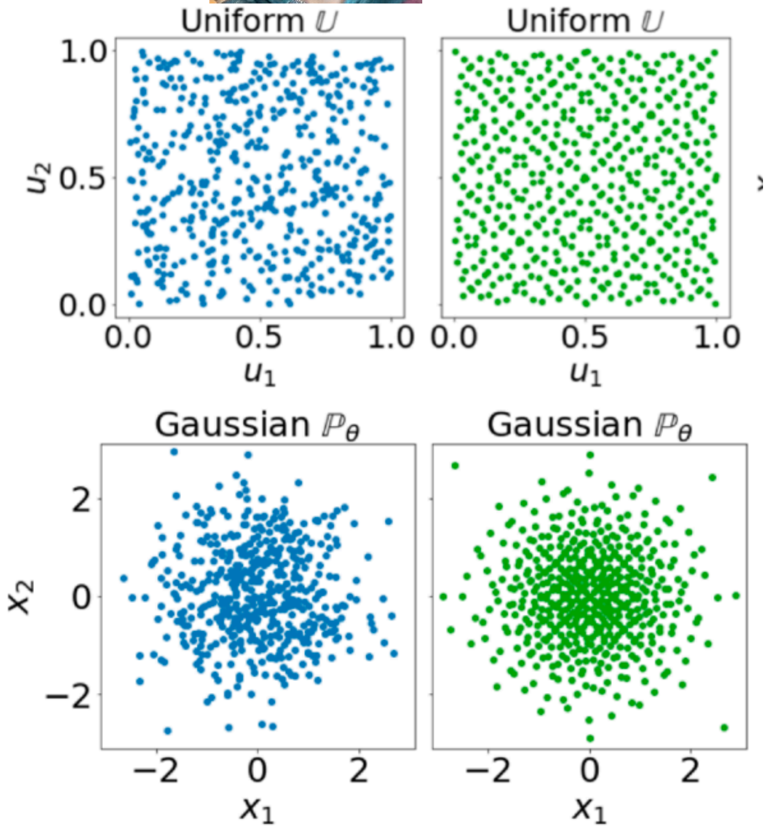

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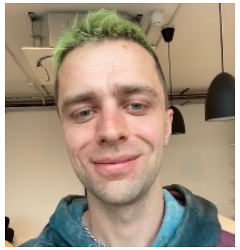
Hypothesis testing for simulator misspecification

H_0: Model/simulator is well-specified.
H_1: Model/simulator is misspecified.

Key, O., Gretton, A., **Briol, F-X.**, & Fernandez, T. (2025). Composite goodness-of-fit tests with kernels. *JMLR*, *26*(51), 1–60.

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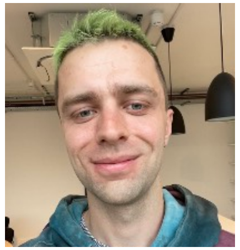

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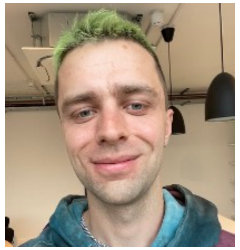

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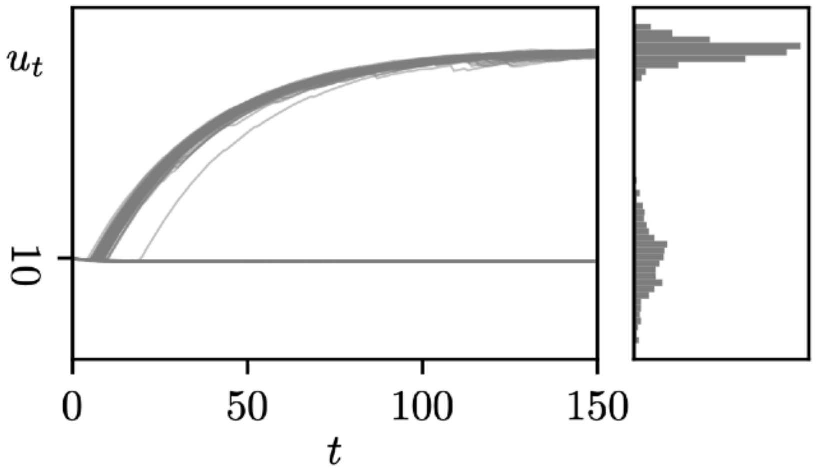
Toggle-switch model:



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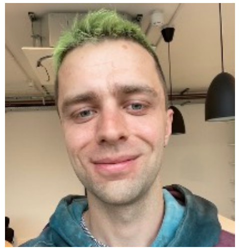

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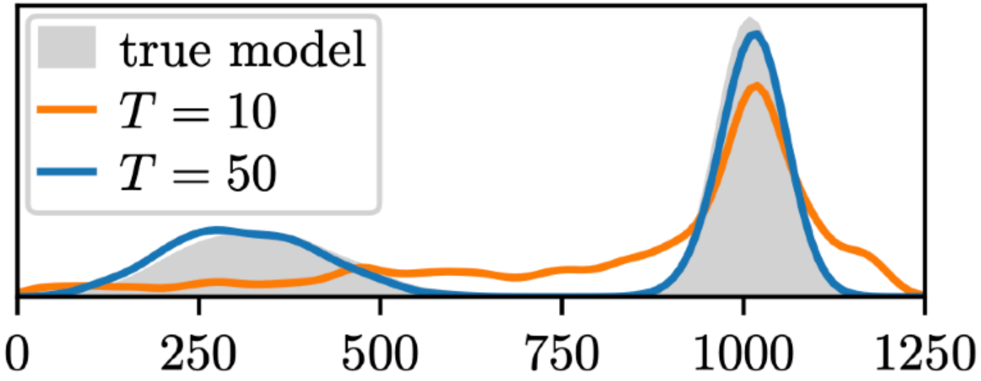

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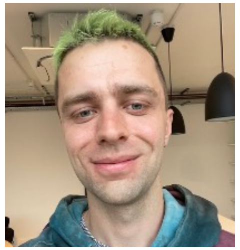

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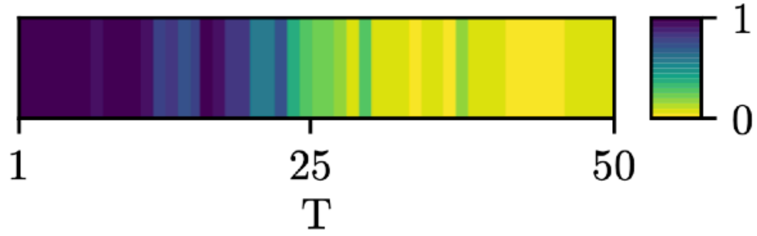
% of rejects:



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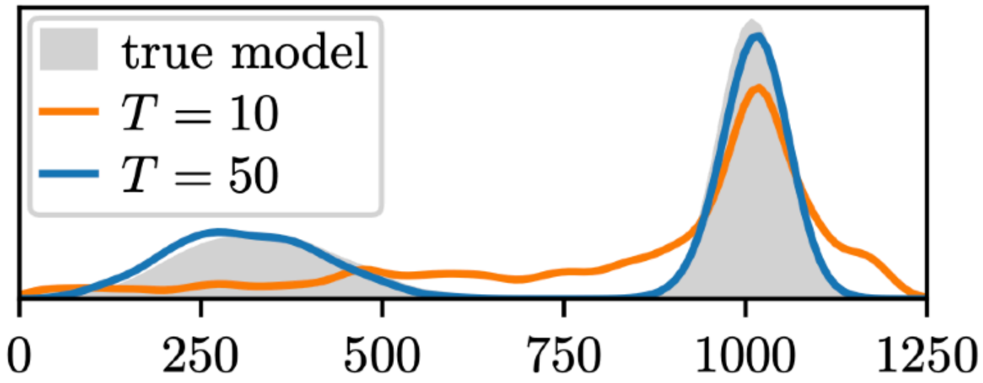

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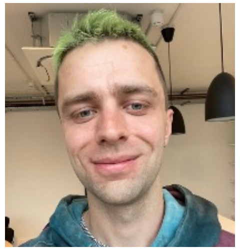
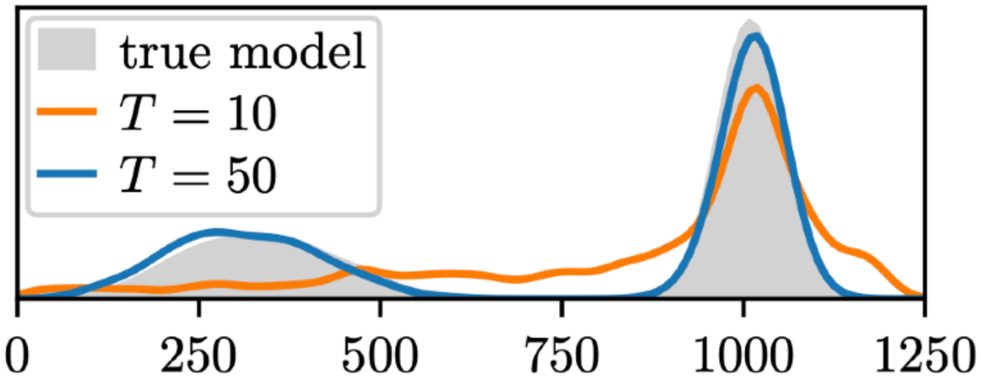
Hypothesis testing for simulator misspecification



H_0: Model/simulator is well-specified.

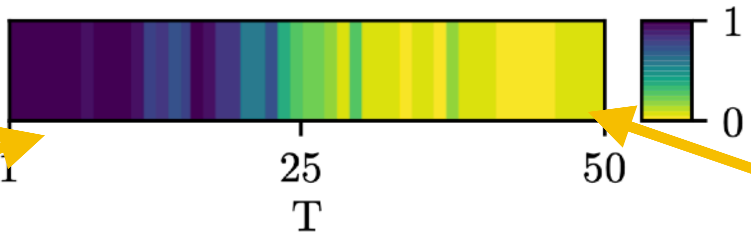
H_1: Model/simulator is misspecified.

Toggle-switch model:



Test statistic: $$\Delta_n = \inf_{\theta \in \Theta} \mathrm{MMD}^2(\mathbb{P}_\theta, \mathbb{Q}_n)$$

% of rejects:



Easy to distinguish

Hard to distinguish

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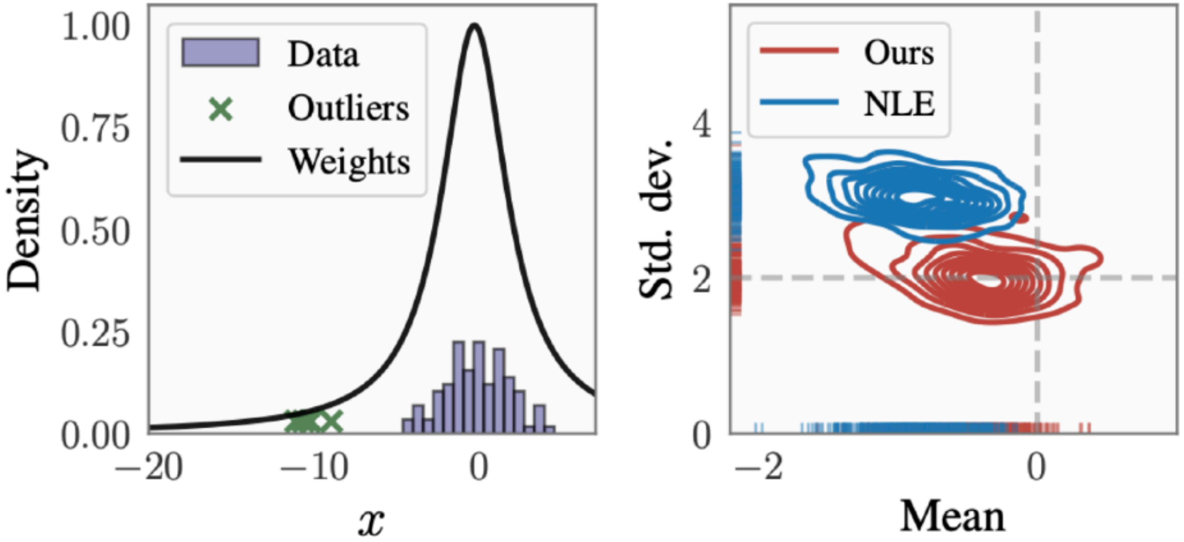

Bharti, A., Dellaporta, H., Hikida, Y., & **Briol, F-X.** (2025+). Amortised and provably-robust simulation-based inference.

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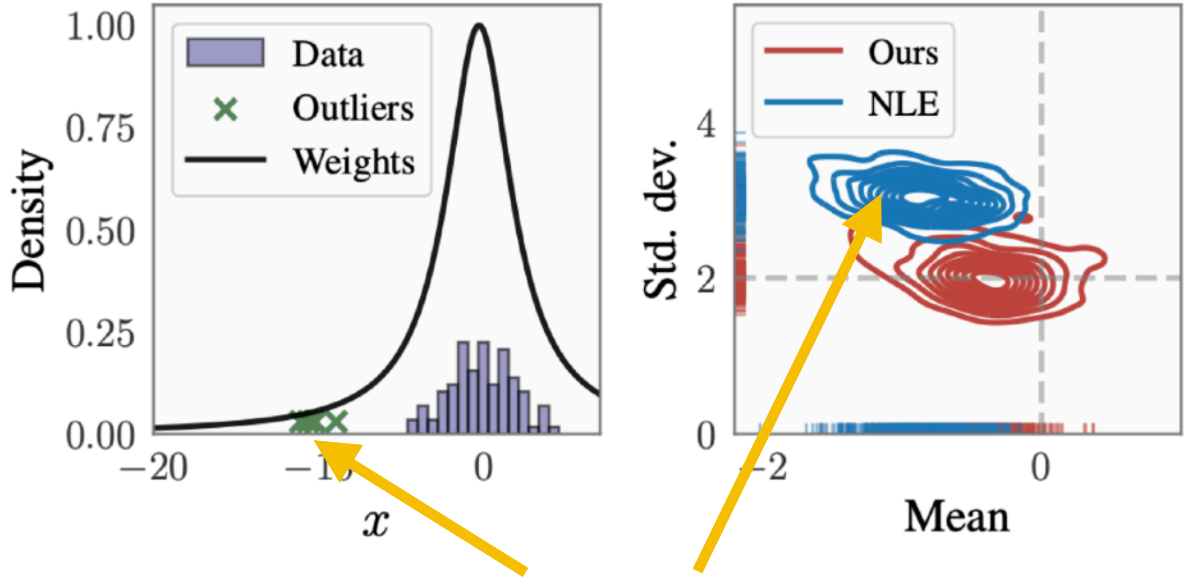

A few outliers can have a drastic impact on NLE!!

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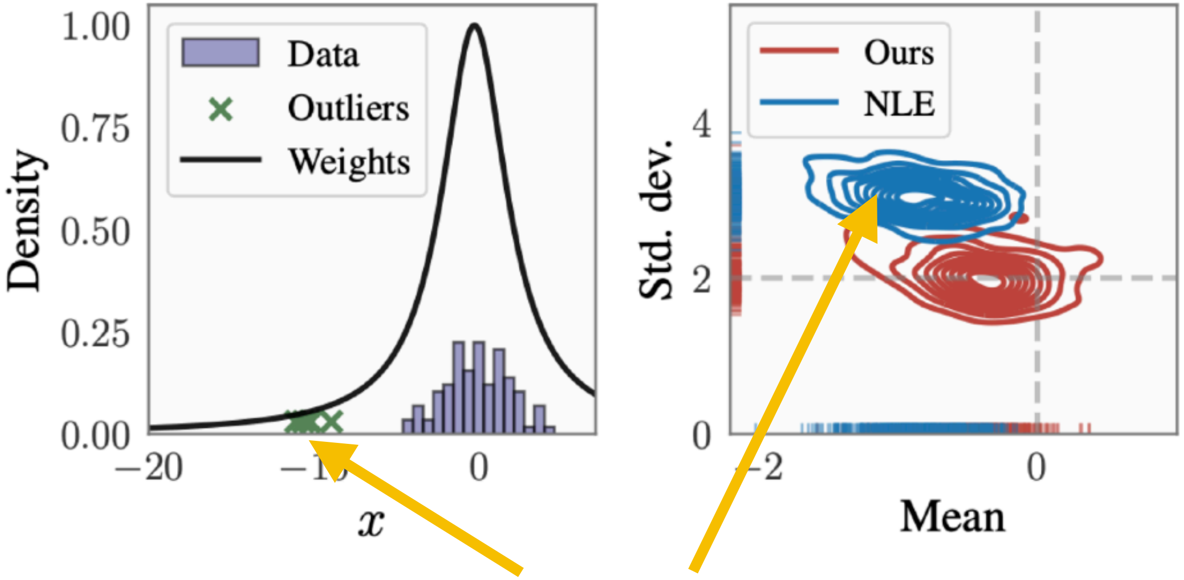

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Looking ahead….

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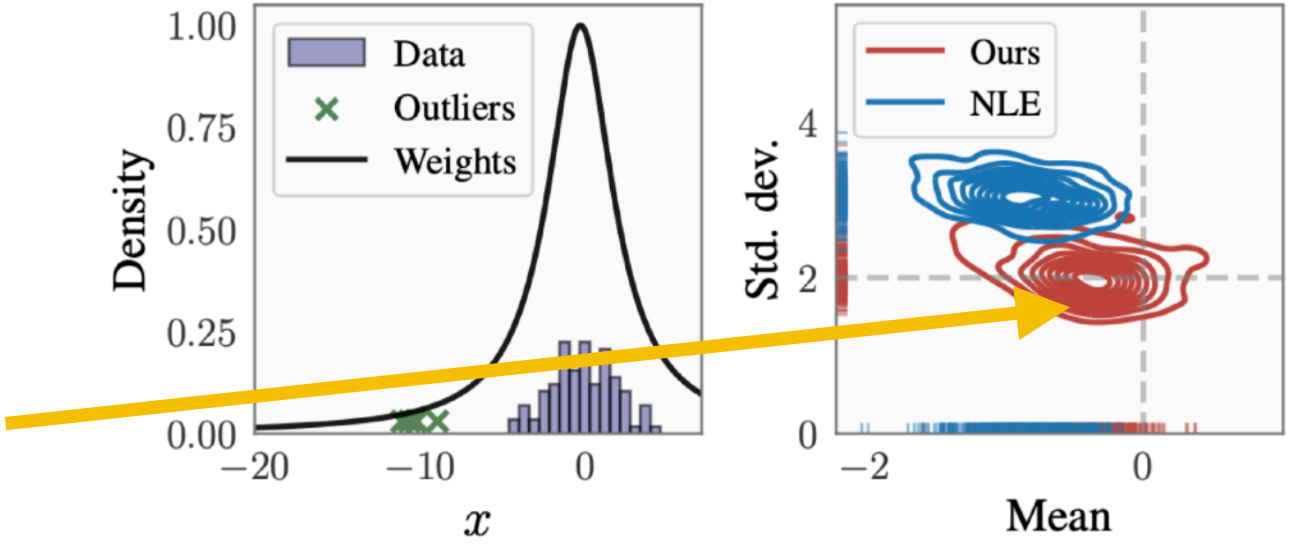
- Currently working on a novel gen-Bayes method to resolve this.



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Any Questions?

Paper: Dellaporta, C., Knoblauch, J., Damoulas, T. & **Briol, F-X** (2022). Robust Bayesian inference for simulator-based models via the MMD posterior bootstrap. AISTATS, 943-970. Best paper award.

Code: https://github.com/haritadell/npl_mmd_project

Summary of this course

- Basic methods:

Minimum distance estimation | Approximate Bayesian Computation | Neural simulation-based inference

- Modern Challenges for SBI (expensive simulators, misspecification, calibration, high-dimensionality).

- Some illustrations of recent advances:

Hikida, Y., Bharti, A., Jeffrey, N. & **Briol, F-X** (2025). Multilevel neural simulation-based inference. arXiv:2506.06087 (to appear at NeurIPS?).

Bharti, A., Huang, D., Kaski, S., & **Briol, F.-X.** (2025). Cost-aware simulation-based inference. International Conference on Artificial Intelligence and Statistics, 28–36.

Dellaporta, C., Knoblauch, J., Damoulas, T. & **Briol, F-X** (2022). Robust Bayesian inference for simulator-based models via the MMD posterior bootstrap. AISTATS, 943-970. Best paper award.

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- **Software:** sbi, bayesflow, etc…

Some personal take-aways

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- Where are the computational statisticians (including me)?!

  → They were sleeping, but are slowly waking up to neural-based methods!